RECOVERY OF EDGES IN SIGNALS AND IMAGES BY MINIMIZING NON-CONVEX OBJECTIVE FUNCTIONS

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Outline

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2. Illustration on $\mathbb{R}$
3. Either Shrinkage or Enhancement of the Differences
   - Smooth and Non-smooth at Zero Potential Functions
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4. Selection for the global minimizer
   - Smooth and Non-smooth at Zero Potential Functions
   - The Truncated Quadratic and the "0-1" Functions
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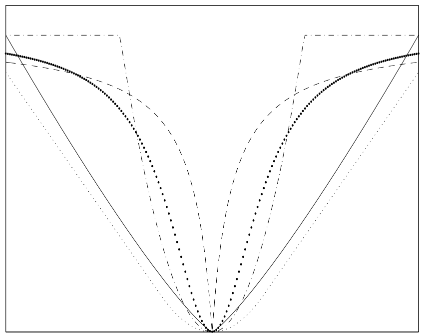
1. Regularized cost-function

Recover signal or image $\hat{\boldsymbol{x}} \in \mathrm{R}^p$ from data $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} +$ noise $\in \mathrm{R}^q$ by minimizing $\boldsymbol{\mathcal{F}}(., \boldsymbol{y})$

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\boldsymbol{x}, \boldsymbol{y}) &= \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|^2 + \beta\boldsymbol{\Phi}(\boldsymbol{x}) \\ \boldsymbol{\Phi}(\boldsymbol{x}) &= \sum_{i \in J} \varphi(\boldsymbol{g}_i^T \boldsymbol{x}), \quad \#J = r \end{aligned}$$

- $\boldsymbol{A}$ linear operator (blur, projection, wavelet, ...), $\boldsymbol{A}^T\boldsymbol{A}$ invertible
- $\{\boldsymbol{g}_i^T\boldsymbol{x} : i \in J\}$: differences between adjacent neighbors

  if $\boldsymbol{x}$ 1-D signal, $\boldsymbol{g}_i^T\boldsymbol{x} = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i$
- $\varphi : \mathbf{R} \to \mathbf{R}_+$ potential function, $\boldsymbol{\Phi}$ regularization, $\beta > \mathbf{0}$ parameter



$\varphi(t) = t^\alpha,\ 0 < \alpha \le 2$ $\qquad \varphi(t) = \sqrt{\alpha + t^2}$

$\varphi(t) = \log(\cosh(t/\alpha))$ $\qquad \varphi(t) = 1 - \exp(-\alpha t^2)$

$\varphi(t) = \alpha t^2/(1 + \alpha t^2)$ $\qquad \varphi(t) = \alpha|t|/(1 + \alpha|t|)$

$\varphi(t) = \min\{\alpha t^2, 1\}$ $\qquad \varphi(t) = \log(\alpha|t| + 1)$

- *Applications:* image restoration, segmentation, motion estimation, color reproduction, optical imaging, tomography, seismic and nuclear imaging, *etc.*

- *Interpretations of $\mathcal{F}$:* $\left\{ \begin{array}{l} \text{Variational and PDE approach [Rudin92,Black96,Weickert98...]} \\ \text{Statistical (Bayesian) methods [Geman85,Besag86,Li95...]} \end{array} \right.$

Our ambition: Catch the essential features exhibited by the (local) minimizers $\hat{\boldsymbol{x}}$ of $\boldsymbol{\mathcal{F}}(.,\boldsymbol{y})$ in connection with the convexity of $\boldsymbol{\Phi}$
Point of interest: the recovery of edges in $\hat{\boldsymbol{x}}$

N.B. *Few theoretical results when φ is non-convex*

$\Rightarrow$ *We analyze the behavior of the (local) minimizers $\hat{\boldsymbol{x}}$ of $\boldsymbol{\mathcal{F}}(.,\boldsymbol{y})$ under variations of data $\boldsymbol{y}$*

$\boxed{\text{Assumptions on } \varphi}$

$$\varphi(t) = \varphi(-t),\ \ \varphi \in \mathcal{C}^2(0,\infty),\ \ \varphi'(t) \geq 0,\ \ \forall t > 0,\ \varphi(0) = 0 \text{ is a strict minimum}$$

$$(\exists)\ \theta > 0\ :\ \varphi''(\theta) < 0 \quad \Rightarrow \quad \varphi''(t) \leq 0,\ \forall t \geq \theta \ \text{ and } \ \lim_{t\to\infty} \varphi''(t) = 0$$

φ is smooth at 0

$$\varphi \text{ is } \mathcal{C}^2,\ \ \exists \tau > 0\ :\ \varphi''(t) \geq 0,\ \ \forall t \in [0,\tau]$$

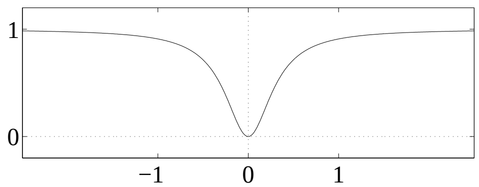
$$\exists \mathcal{T} > \tau\ :\ \ \varphi'' \begin{cases} \text{decreasing on} & [\tau, \mathcal{T}] \\ \text{increasing on} & [\mathcal{T}, \infty) \end{cases}$$

φ is nonsmooth at 0

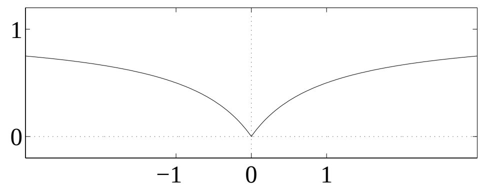
$$\varphi'(0^+) > 0$$

$$\varphi'' \ \text{ increasing on } \ [0,\infty)$$

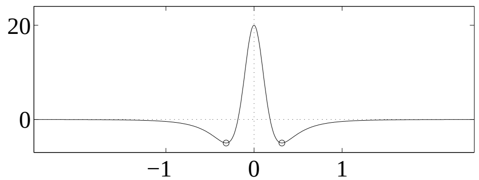
$\varphi(t) = \alpha t^2/(1+\alpha t^2)$



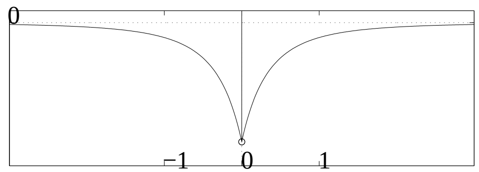
$\varphi(t) = \alpha|t|/(1+\alpha|t|)$



$\varphi''(t)$

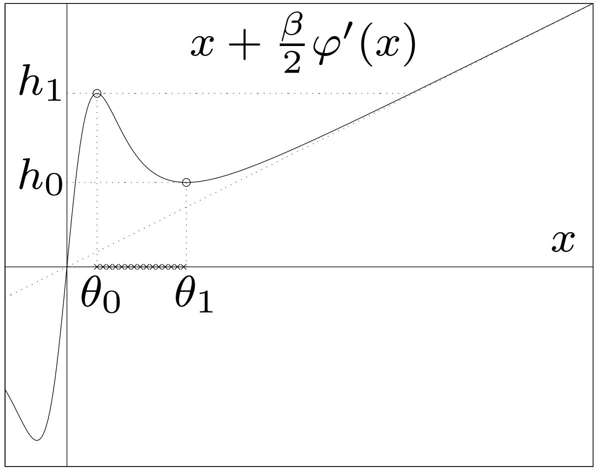


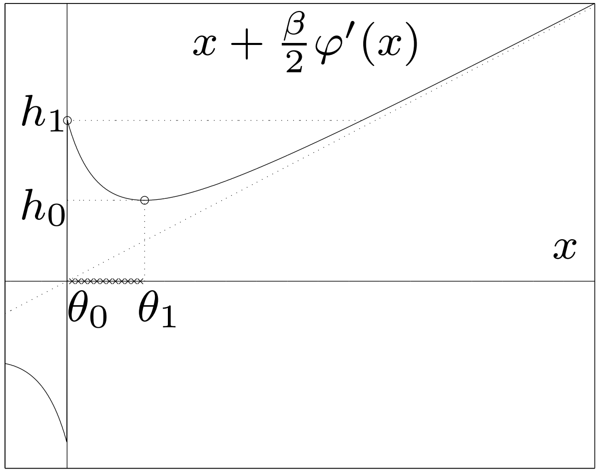
$\varphi''(t)$



2. Illustration on R

$$\mathcal{F}(x,y) = (x-y)^2 + \beta\varphi(x), \quad x, y \in \mathbf{R}$$





$\varphi \sim \mathcal{C}^2$ $\qquad\qquad$ $\varphi'(0^+) > 0$

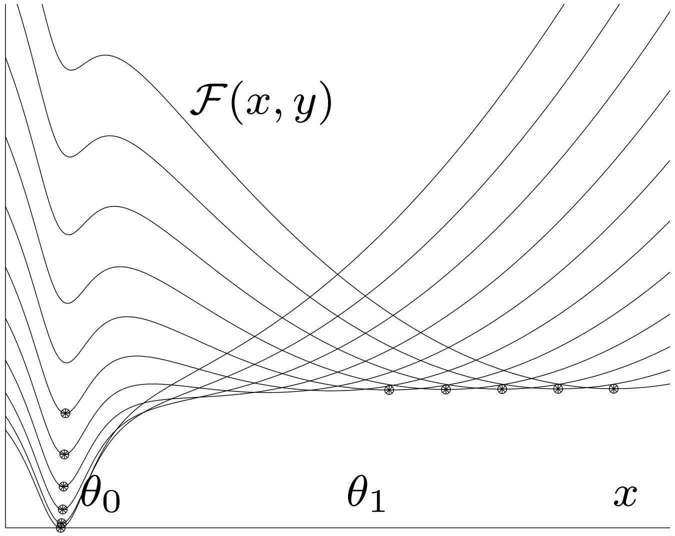
$$(\theta_0, \theta_1) = \left\{t \in (0,\infty) : \varphi''(t) \leq -2/\beta\right\}$$

$\mathcal{F}_y''(x) < 0, \quad \forall x \in (\theta_0, \theta_1) \;\Rightarrow\;$ **No local minimizer lies in (θ_0, θ_1) for any $y > 0$**

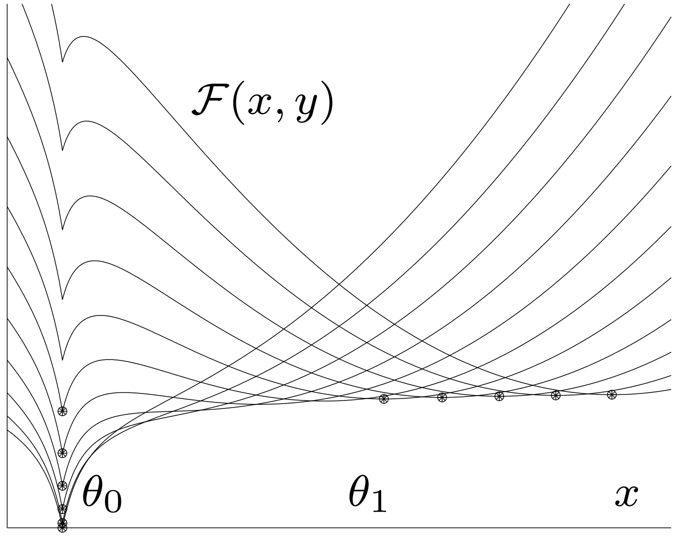
$$h_0 = \theta_1 + \frac{\beta}{2}\varphi'(\theta_1) \;\text{ and }\; h_1 = \theta_0 + \frac{\beta}{2}\varphi'(\theta_0^+)$$

- $|y| \leq h_1 \quad \Rightarrow \quad$ local minimizer $|\hat{x}_0| \leq \theta_0 \quad (= 0 \;\text{ if }\; \varphi'(0^+) > 0)$ **(strong smoothing)**
- $|y| \geq h_0 \quad \Rightarrow \quad$ local minimizer $|\hat{x}_1| \geq \theta_1$ **(loose smoothing)**

- $\exists h \in (h_0, h_1)$ $\quad \begin{array}{lll} |y| \le h & \Rightarrow & \text{the global min} = \hat{x}_0 \\ |y| \ge h & \Rightarrow & \text{the global min} = \hat{x}_1 \end{array}$ $\quad \begin{array}{l} \textbf{(strong smoothing)} \\ \textbf{(loose smoothing)} \end{array}$



$\mathcal{F}(x,y) = (x-y)^2 + \beta \frac{\alpha x^2}{(1+\alpha x^2)}$



$\mathcal{F}(x,y) = (x-y)^2 + \beta \frac{\alpha |x|}{(1+\alpha |x|)}$

Each curve $=\mathcal{F}(x,y)$ for a different y

For $\boldsymbol{y = h}$ the global minimizer jumps from $\boldsymbol{\hat{x}_0}$ to $\boldsymbol{\hat{x}_1}$

$\equiv$ **decision on the degree of smoothing**

Main Results

I. $\exists\ \boldsymbol{\theta_0 > 0}$ and $\boldsymbol{\theta_1 > \theta_0}$ (strictly)

$\hat{\boldsymbol{x}} =$ (local) minimizer of $\boldsymbol{\mathcal{F}_y}$ $\quad\Rightarrow\quad$ $\boxed{\text{either } |\boldsymbol{g_i^T \hat{x}}| \leq \boldsymbol{\theta_0} \text{ or } |\boldsymbol{g_i^T \hat{x}}| \geq \boldsymbol{\theta_1}, \quad \forall \boldsymbol{i} \in \boldsymbol{J}}$

If $\boldsymbol{\varphi'(0^+) > 0}$ then $\boldsymbol{\theta_0 = 0}$

$$\widehat{J}_0 = \left\{i \in J : \left|g_i^T \hat{x}\right| \leq \theta_0\right\} \qquad \text{and} \qquad \widehat{J}_1 = \left\{i \in J : \left|g_i^T \hat{x}\right| \geq \theta_1\right\}$$

the homogeneous regions $\qquad\qquad$ the edges

If $\varphi'(0^+) > 0$ and $\{g_i\}$—1^{st}-order differences then homogeneous regions are constant

II. $\{g_i\}$—1^{st}-order differences

$$\boldsymbol{1\!\!1_\Sigma[i]} = \begin{cases} \mathbf{1} & \text{if } \boldsymbol{i \in \Sigma} \\ \mathbf{0} & \text{if } \boldsymbol{i \in \Omega \setminus \Sigma} \end{cases} \qquad \begin{array}{lcl} \text{Original} & : & \boldsymbol{h 1\!\!1_\Sigma,\ h > 0} \\ \text{Data} & : & \boldsymbol{y = hA1\!\!1_\Sigma} \end{array} \qquad \hat{\boldsymbol{x}} = \textbf{global} \text{ minimizer of } \boldsymbol{\mathcal{F}_y}$$

$\exists\ \boldsymbol{h_0 > 0}$ and $\boldsymbol{h_1 > h_0}$

$$\begin{array}{lcl} \boldsymbol{h \in (0, h_0)} & \Rightarrow & |\boldsymbol{g_i^T \hat{x}}| \leq \boldsymbol{\theta_0}, \quad \forall \boldsymbol{i} \in \boldsymbol{J} \\ \boldsymbol{h > h_1} & \Rightarrow & \boldsymbol{\hat{x} \approx h 1\!\!1_\Sigma} \ \ (|g_i^T \hat{x}| \geq \theta_1, \quad \forall i \in J_1) \end{array}$$

$\boldsymbol{\exists \zeta > 0\ :\ \mathcal{F}_y(\hat{x}) \leq \zeta, \quad \forall h > 0}$

3. Either Shrinkage or Enhancement of the Differences

3.1 Smooth at Zero PFs

Theorem 1 $\mathcal{F}(x, y) = \|Ax - y\|^2 + \beta \sum_i \varphi(g_i^T x)$

- φ *is* $\mathcal{C}^2$ *and nonconvex as assumed*
- $\{g_i\}$ *is linearly independent and* $\mu = \max_{i \in J} \|G^T (GG^T)^{-1} e_i\|$

(i) $\beta > \frac{2\mu^2 \, \|A^T A\|}{|\varphi''(\mathcal{T})|} \Rightarrow \exists \;\; \theta_0 \in (\tau, \mathcal{T})$ *and* $\theta_1 > \mathcal{T}$:

$$\boxed{\mathcal{F}_y \text{ has a (local) min. at } \hat{x} \Rightarrow \text{either } |g_i^T \hat{x}| \leq \theta_0 \text{ or } |g_i^T \hat{x}| \geq \theta_1, \; \forall i \in J \quad (\star)}$$

(ii) $\theta_0 \in (\tau, \mathcal{T})$, $\varphi''(\theta_0) < 0 \quad \Rightarrow \quad \exists \theta_1 > \mathcal{T}, \; \exists \beta_1$ *such that* $[\; \beta > \beta_1 \quad \Rightarrow \quad (\star)$ *holds* $]$

The same holds if we interchange θ_0 *and* θ_1

(iii) $\theta_1 - \theta_0$ *increases with* β

Proof: if $|g_i^T \hat{x}| \in (\theta_0, \theta_1)$ then $\exists u \in \mathrm{R}^p$ such that $D_1^2 \mathcal{F}(\hat{x}, y)(u, u) < 0 \Rightarrow$ no minimum at $\hat{x}$

$\Rightarrow$ Thresholds θ_0 and θ_1 are pessimistic

("true θ_0" ¡ θ_0 and "true θ_1" ¿θ_1)

$\{g_i^T\}$ linearly independent if x is a signal

Difficult to extend to images (the directions where $D_1^2 \mathcal{F}(\hat{x}, y)(u, u) < 0$ depend on $\hat{x}$)

Conjecture that remains true: for $g_i^T \hat{x} \approx \mathcal{T}, \quad \mathcal{F}_y$ is likely to be concave there

$\boxed{\textit{Truncated Quadratic PF}\ \ \boldsymbol{\varphi(t) = \min\{\alpha t^2,\ 1\}}}$

Notations: $\quad P = I - \dfrac{1}{\|A1\!\!1\|^2} A1\!\!1(A1\!\!1)^T \qquad u_i[j] = \begin{cases} 1 & \text{if} \quad j = 1, \dots, i \\ 0 & \text{if} \quad j = i+1, \dots, p, \end{cases} \qquad \forall i = 1, \dots, p$

Theorem $\qquad \mathcal{F}(x, y) = \|Ax - y\|^2 + \beta \sum_{i=1}^{p-1} \varphi(x_{i+1} - x_i)$

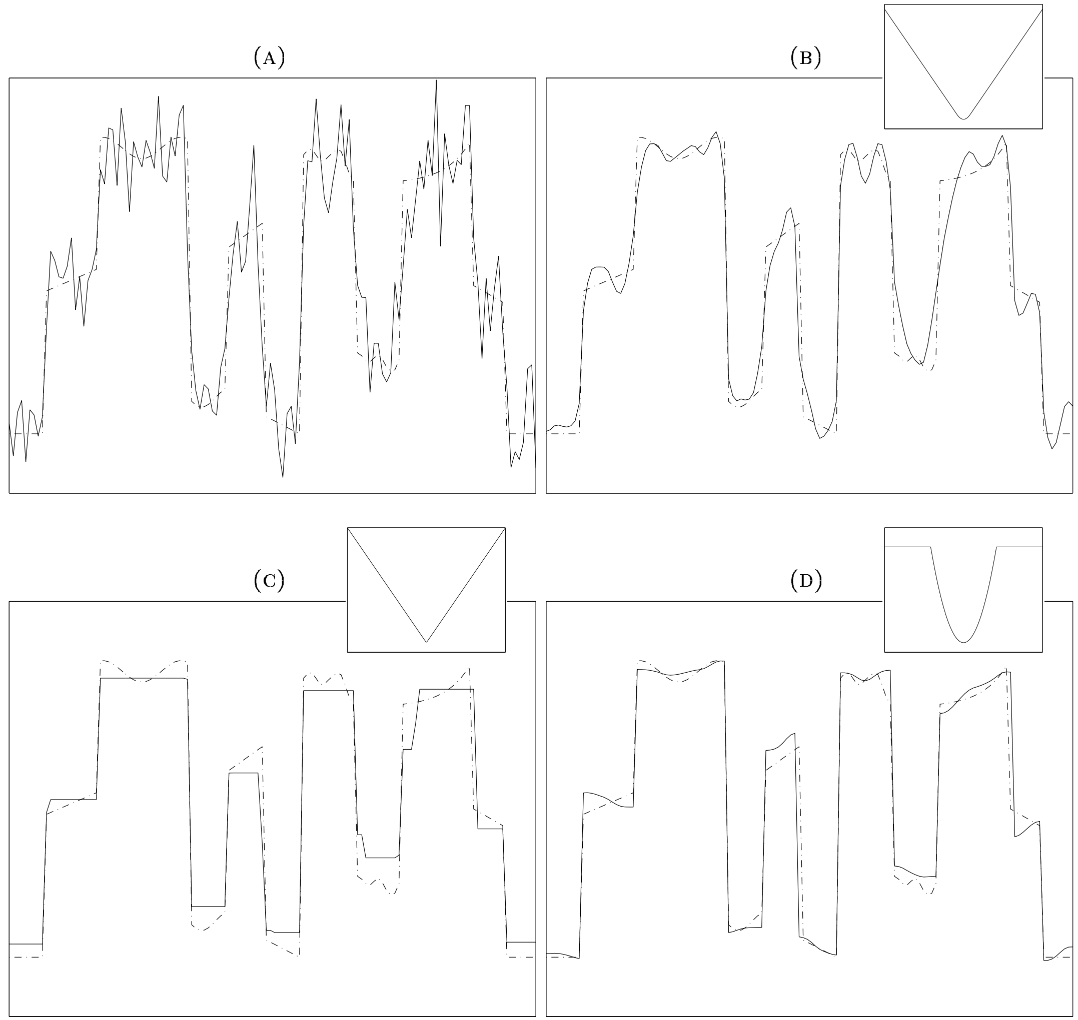
$\boldsymbol{\mathcal{F}(.,y)}$ *has a global minimum at* $\boldsymbol{\hat{x}} \quad \Rightarrow \quad \forall \boldsymbol{i = 1, \dots, p-1}$

- $\boldsymbol{PAu_i = 0} \quad \Rightarrow \quad \boldsymbol{\hat{x}_{i+1} = \hat{x}_i}$
- $\boldsymbol{PAu_i \neq 0} \quad \Rightarrow \quad \boxed{\textit{either}\ |\boldsymbol{\hat{x}_{i+1} - \hat{x}_i}| \leq \dfrac{\boldsymbol{\Gamma_i}}{\boldsymbol{\sqrt{\alpha}}} \quad \textit{or} \quad |\boldsymbol{\hat{x}_{i+1} - \hat{x}_i}| \geq \dfrac{\boldsymbol{1}}{\boldsymbol{\sqrt{\alpha}\Gamma_i}}}$

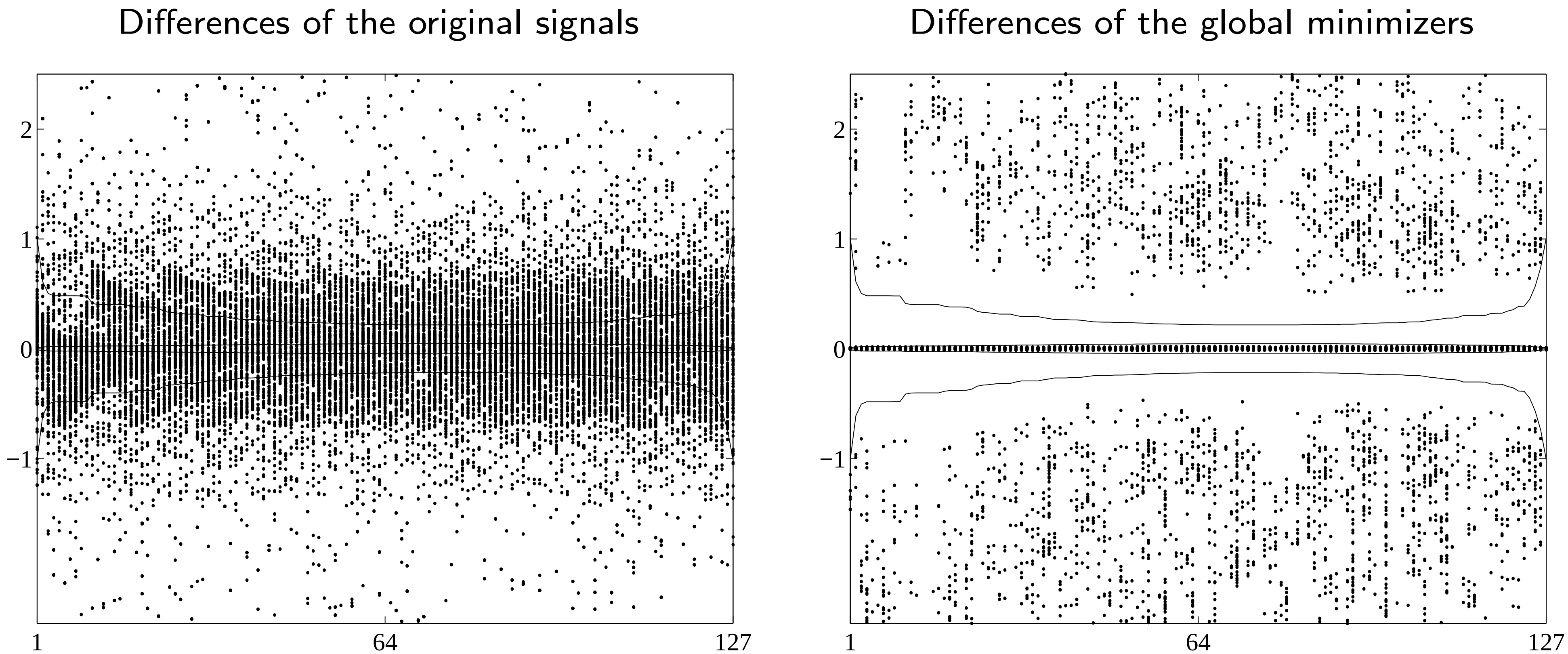
$\boldsymbol{\Gamma_i = \sqrt{\frac{\|PAu_i\|^2}{\alpha\beta + \|PAu_i\|^2}} < 1}$ *(independent of data* $\boldsymbol{y}$*)*

Strict inequalities if unique global minimizer

- Bounds adapted to each difference
- Connection with Theorem 1 : $\theta_0 = \dfrac{\gamma}{\sqrt{\alpha}} < \theta_1 = \dfrac{1}{\sqrt{\alpha}\gamma}$ for $\gamma = \max_{i \in J} \Gamma_i$
- Can fail if $\hat{x}$ is local $\neq$ global minimizer
- Necessary condition for $\hat{x}$ to be global minimzier



(A) Noisy data $y = x * h + n$ with $h_k = \exp^{-\frac{0.4k^2}{1.4}}$, $|k| \leq 5$ and n white Gaussian noise, 10 dB SNR.



Distribution of the differences for 100 signals.

Thresholds $\pm\Gamma_i/\sqrt{\alpha}$, $\pm 1/\sqrt{\alpha}\Gamma_i$ for $i = 1, \ldots, 127$ (—). X-axis: positions of differences $i = 1, \ldots, 127$.

Y-axis: a dot at position i is the value of the ith difference of a signal.

3.2 Non-Smooth at Zero PFs

Theorem 2 $\mathcal{F}(x,y) = \|Ax-y\|^2 + \beta\sum_i \varphi(g_i^T x)$

$\varphi'(0^+) > 0$ *and nonconvex as assumed*

(i) $\beta > \frac{2\mu^2 \ \|A^T A\|}{|\varphi''(0^+)|} \quad \Rightarrow \quad \exists\ \theta_1 > 0$:

$$\boxed{\mathcal{F}_y \text{ has a (local) min. at } \hat{x} \quad \Rightarrow \quad \text{either } |g_i^T\hat{x}| = 0 \ \text{ or } \ |g_i^T\hat{x}| \geq \theta_1, \ \forall i \in J} \quad (\star)$$

(ii) $\theta_1 > 0,\ \varphi''(\theta_1) < 0 \ \Rightarrow \ \exists\beta_1$ *such that* $[\ \beta > \beta_1 \ \Rightarrow \ (\star)$ *holds* $]$

θ_1 *increases with* β

(iii) A^TA *invertible.* $\ \exists\eta > 0 \ : \ \|y\| < \eta \ \Rightarrow \ \widehat{J}_0 = \left\{i \in J : g_i^T\hat{x} = 0\right\} \neq \emptyset$

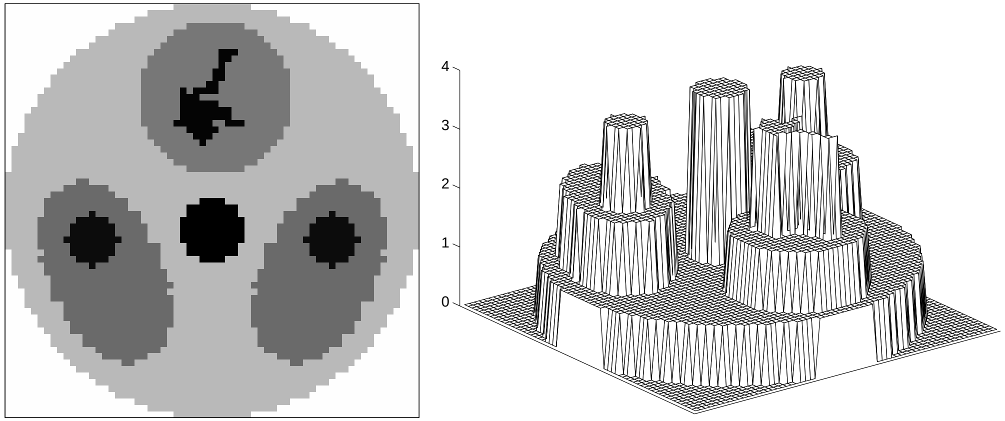
The θ_1 exhibited in the proof is pessimistic

Strong result with no special assumptions

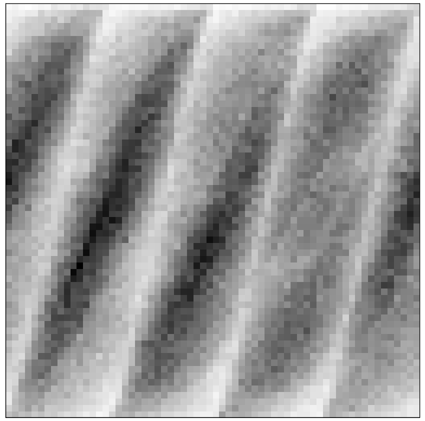
Neat segmentation

N.B. Enhanced stair-casing effect !

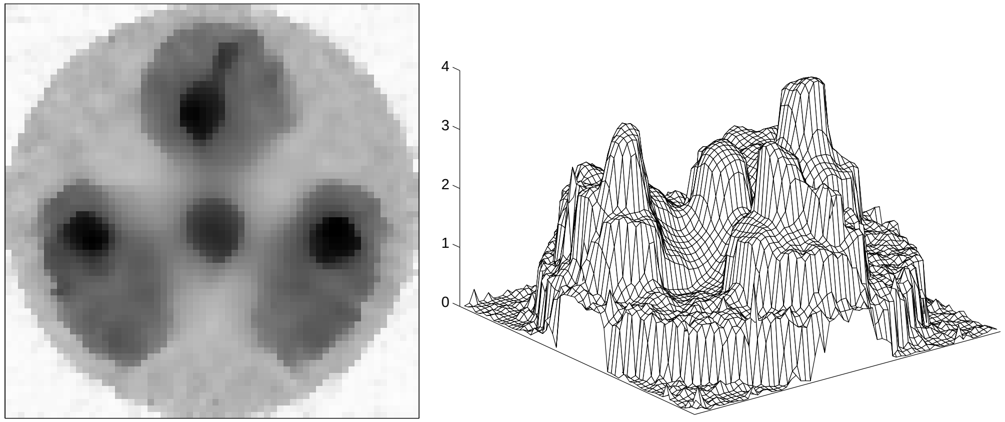
Image Reconstruction in Emission Tomography



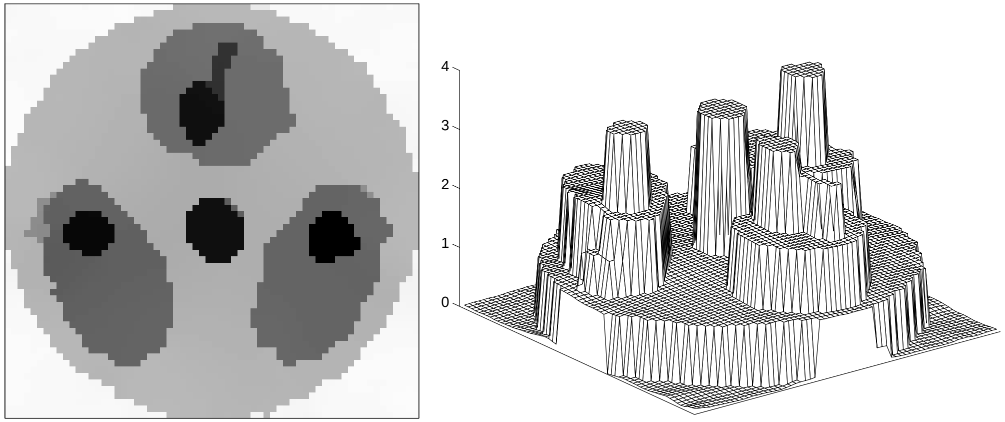
Original phantom



Emission tomography simulated data



φ is smooth (Huber function)



$\varphi(t) = t/(\alpha + t)$ (non-smooth, non-convex)

Reconstructions by minimizing $\mathcal{F}(x, y) = \Psi(x, y) + \beta \sum_{i \sim j} \varphi(|x_i - x_j|)$, $\quad \Psi =$ smooth, convex

$\boxed{\text{"0-1" } PF \quad \boldsymbol{\varphi(0) = 0} \text{ and } \boldsymbol{\varphi(t) = 1} \text{ if } \boldsymbol{t \neq 0}}$

Theorem $\mathcal{F}(x,y) = \|Ax - y\|^2 + \beta \sum_{i=1}^{p-1} \varphi(x_{i+1} - x_i)$

$\boldsymbol{\mathcal{F}(.,y)}$ *has a global minimum at* $\boldsymbol{\hat{x}} \quad \Rightarrow \quad \boldsymbol{\forall i = 1, \ldots, p-1}$

- $\boldsymbol{PAu_i = 0} \quad \Rightarrow \quad \boldsymbol{\hat{x}_{i+1} = \hat{x}_i}$
- $\boldsymbol{PAu_i \neq 0} \quad \Rightarrow \quad \boxed{\text{either } \quad \boldsymbol{\hat{x}_{i+1} = \hat{x}_i} \quad \text{or} \quad \boldsymbol{|\hat{x}_{i+1} - \hat{x}_i| \geq \frac{\sqrt{\beta}}{\|PAHe_i\|}}}$

  Strict inequality if unique global minimizer

- Bounds adapted to each difference
- Connection with Theorem 2 : $\theta_1 = \min_{i \in J} \dfrac{\sqrt{\beta}}{\|PAHe_i\|}$
- Can fail if $\hat{x}$ is local $\neq$ global minimizer
- Necessary condition for $\hat{x}$ to be global minimizer

4. Selection for the Global Minimizer

Additional assumptions

- $\varphi(t) \le 1 \; \forall \, t \in \mathbf{R}$
- $\{g_i\}$—1^{st}-order differences: $g_i^T x = \gamma_i (x_{i_1} - x_{i_2})$ (usually $\gamma_i = 1$)
- $A^T A$ invertible ($\Rightarrow$ $\exists$ global minimizer)

$$\mathbb{1}_{\Sigma}[i] = \begin{cases} 1 & \text{if } \; i \in \Sigma \\ 0 & \text{if } \; i \in \Omega \setminus \Sigma \end{cases} \qquad\qquad \begin{array}{l} J_1 = \{ i \in J : |g_i^T \mathbb{1}_{\Sigma}| \neq 0 \} \\ J_0 = J \setminus J_1 \end{array}$$

Ω the domain of x, $\Sigma \subset \Omega$ connected w.r.t $\{g_i\}$

- Original image or signal $= h \mathbb{1}_{\Sigma}$, $h > 0$ (the scaled characteristic function of Σ)
- Data $y = h A \mathbb{1}_{\Sigma}$

$\Rightarrow$ Characterize the global minimizers $\hat{x}$ of $\mathcal{F}_y$

For every $y \in \mathbf{R}^q$ we denote by $\hat{x}$ a global minimizer of $\mathcal{F}_y$

4.1 Smooth at Zero PFs

Theorem $\mathcal{F}(x,y) = \|x-y\|^2 + \beta \sum_i \varphi(g_i^T x)$

$A = I$, $\varphi \in \mathcal{C}^2$ *(all assumptions)*, $\beta > \beta_0$, (θ_0, θ_1) *as in Theorem 1*

$$y = hA\mathbb{1}_\Sigma, \quad h > 0 \qquad (*)$$

(i) $\exists h_0 > 0 : h \in (0, h_0) \quad \Rightarrow \quad |g_i^T \hat{x}| \le \theta_0, \quad \forall i \in J$

(ii) $\exists h_1 > 0, \exists \sigma > 0 :$

$$h > h_1 \quad \Rightarrow \quad \begin{array}{ll} |g_i^T \hat{x}| \le \beta\sigma & \forall i \in J_0 \\ |g_i^T \hat{x}| \ge \max\left\{ h|g_i^T \mathbb{1}_\Sigma| - \beta\sigma,\ \beta\sigma,\ \theta_1 \right\} & \forall i \in J_1 \end{array}$$

Moreover: $\exists\ \boldsymbol{\zeta > 0}$ *such that* $\boldsymbol{\mathcal{F}_y(\hat{x}) \le \zeta}$, $\ \forall \boldsymbol{h > 0}$ *in* $(*)$

- All differences $g_i^T \hat{x}$ for $i \in J_0$ remain bounded by the same constant for all $h > h_1$
- Constants in the proof are pessimistic
- ζ is easy to compute, necessary condition for global minimum

$\boxed{\textit{Truncated Quadratic PF}\ \ \boldsymbol{\varphi(t) = \min\{\alpha t^2,\ 1\}}}$

Proposition $\qquad \mathcal{F}(x,y) = \|Ax - y\|^2 + \beta \sum_{i\in J} \varphi(g_i^T x)$

$A^T A$ invertible

$$\boldsymbol{y = hA\mathbb{1}_\Sigma, \quad h > 0}$$

$$\boldsymbol{\chi_\Sigma = \left(A^T A + \beta\alpha G^T G\right)^{-1} A^T A\mathbb{1}_\Sigma} \quad \text{(regularized least-squares for } h = 1\text{)}$$

$\boldsymbol{\exists h_0 > 0, \quad \exists h_1 > h_0}$ such that

(i) $\boldsymbol{h \in (0, h_0) \quad \Rightarrow \quad \hat{x} = h\ \chi_\Sigma}$

(ii) $\boldsymbol{h > h_1 \quad \Rightarrow \quad \hat{x} = h\ \mathbb{1}_\Sigma}$

$\boldsymbol{\hat{x}}$ for $\boldsymbol{h \in (0, h_0)}$ does not contain edges

4.2 Non-Smooth at Zero PFs

Theorem $\mathcal{F}(x,y) = \|x-y\|^2 + \beta \sum_i \varphi(g_i^T x)$

$\boldsymbol{\varphi'(0^+) > 0}$, *nonconvex as assumed, bounded,* $A = I$ *and* $g_i^T x = x_{i_1} - x_{i_2}$

$\beta > \beta_0$ *and* θ_1 *as in Theorem 2*

$$\boldsymbol{y = h\, \mathbb{1}_{\Sigma}, \quad h > 0} \tag{$*$}$$

(i) $\exists h_0 > 0 \ : \ \boldsymbol{h \in (0, h_0)} \quad \Rightarrow \quad \boldsymbol{\hat{x} = h \frac{\#\Sigma}{p} \mathbb{1}}$

(ii) *If* $\varphi(\theta_1) \geq (\#J - 2)/(\#J - 1), \quad \exists h_1 > 0 \ : \quad \boldsymbol{h > h_1} \quad \Rightarrow \quad \boldsymbol{\hat{x} = \hat{s}\mathbb{1}_{\Sigma} + \hat{c}\mathbb{1}}$

$\hat{s} \leq h$ *and* $\hat{c} \geq 0$

$\hat{s} \to h$ *and* $\hat{c} \to 0$ *as* $h \to \infty$

Moreover: $\exists \zeta > 0$ *such that* $\boldsymbol{\mathcal{F}_y(\hat{x}) \leq \zeta, \quad \forall h > 0}$ *in* $(*)$

- $\hat{x} \to h\mathbb{1}_{\Sigma}$ as $h \to \infty$
- Constants in the proof are pessimistic
- ζ is easy to compute, necessary condition for global minimum

"0-1" PF $\boldsymbol{\varphi(0) = 0}$ *and* $\boldsymbol{\varphi(t) = 1}$ *if* $\boldsymbol{t \neq 0}$

Proposition $\mathcal{F}(x,y) = \|Ax - y\|^2 + \beta \sum_{i \in J} \varphi(g_i^T x)$

$A^T A$ invertible

$$\boldsymbol{y = hA\mathbb{1}_\Sigma, \quad h > 0}$$

$\exists \boldsymbol{h_0 > 0}, \quad \exists \boldsymbol{h_1 > h_0}$ such that

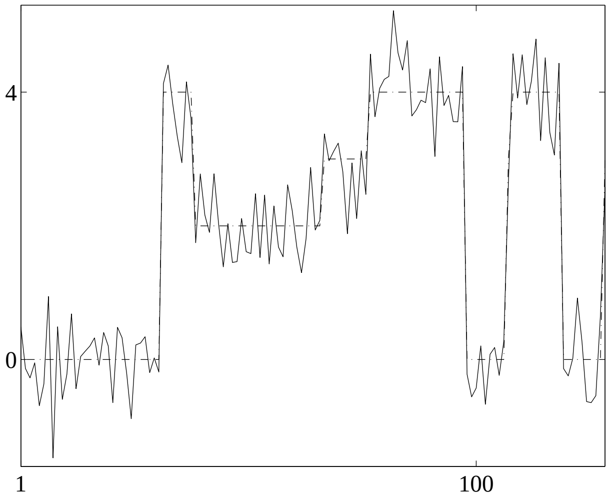
(i) $\boldsymbol{h \in (0, h_0)} \quad \Rightarrow \quad \boldsymbol{\hat{x} = h\ \hat{c}\ \mathbb{1}}$ for $\boldsymbol{\hat{c} = \frac{(A\mathbb{1})^T (A\mathbb{1}_\Sigma)}{\|A\mathbb{1}\|^2}}$

(ii) $\boldsymbol{h > h_1} \quad \Rightarrow \quad \boldsymbol{\hat{x} = h\ \mathbb{1}_\Sigma}$

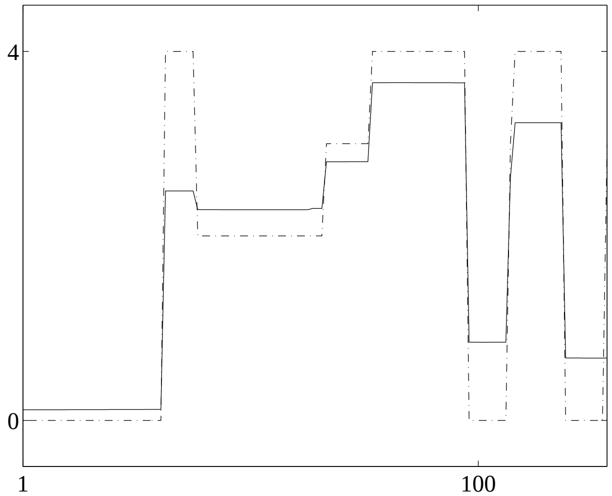
Lemma

$\hat{x} \in \mathrm{R}^p$ is a local minimizer of $\mathcal{F}_y$ $\Leftrightarrow$ $\left[\begin{array}{l} \hat{x} \text{ is a solution to the problem:} \\ \text{minimize } \|Ax - y\|^2 \text{ subject to} \\ \qquad x \in \{u \in \mathrm{R}^p : g_i^T u = 0, \ \forall i \in \widehat{J}_0\} \\ \qquad \widehat{J}_0 = \{i \in J : g_i^T \hat{x} = 0\} \end{array}\right.$

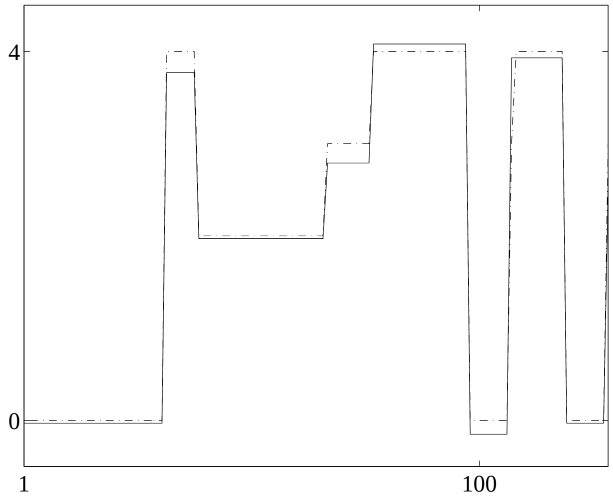
5. Comparison with Convex Edge-Preserving Regularization



Data $y = x + n$



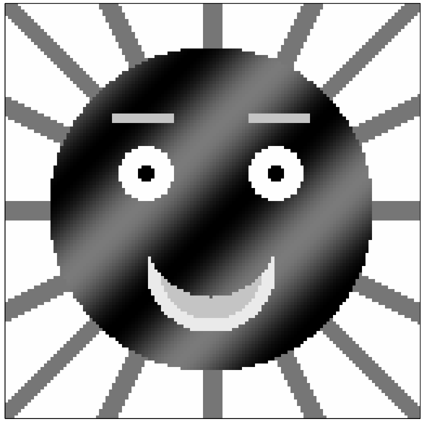
$\varphi(t) = |t|$



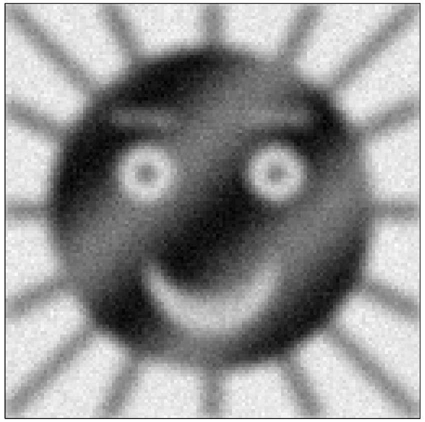
$\varphi(t) = \alpha|t|/(1 + \alpha|t|)$

- If $\mathcal{F}_y$ is convex, then differences at the minimum $g_i^T \hat{x}$ can take any value on R
- Convex edge-preserving regularization (TV) creates strongly homogeneous regions (because non-smooth at zero), a fortiori these are separated by sharp transitions (edges) whose amplitude is attenuated
- Edge-detection using φ non-convex is fundamentally different : it relays on the classification of the differences and the jumps of the global minimizer between local minimizers corresponding to different configurations for the edges

6. Illustration



Original image

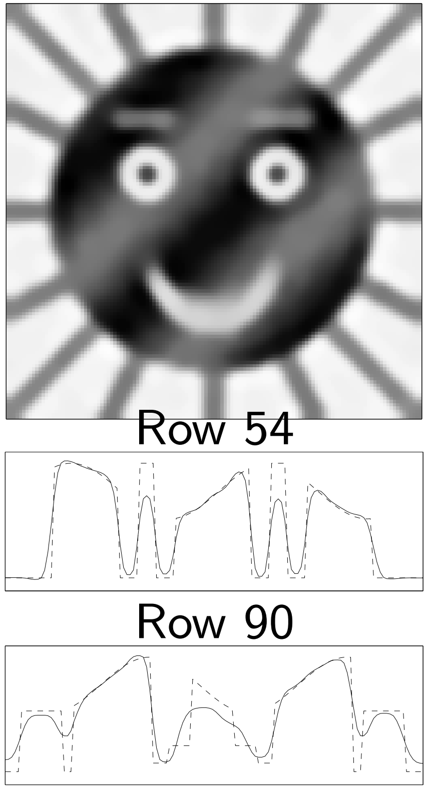


Data y

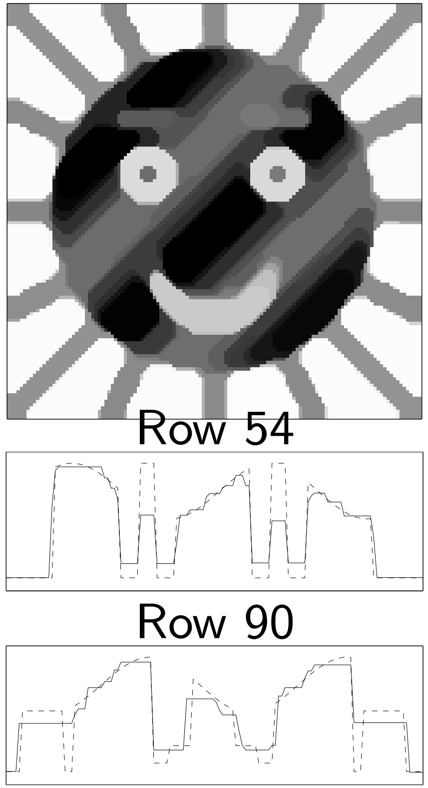
Data $y = a \star x + n$, a is blur,

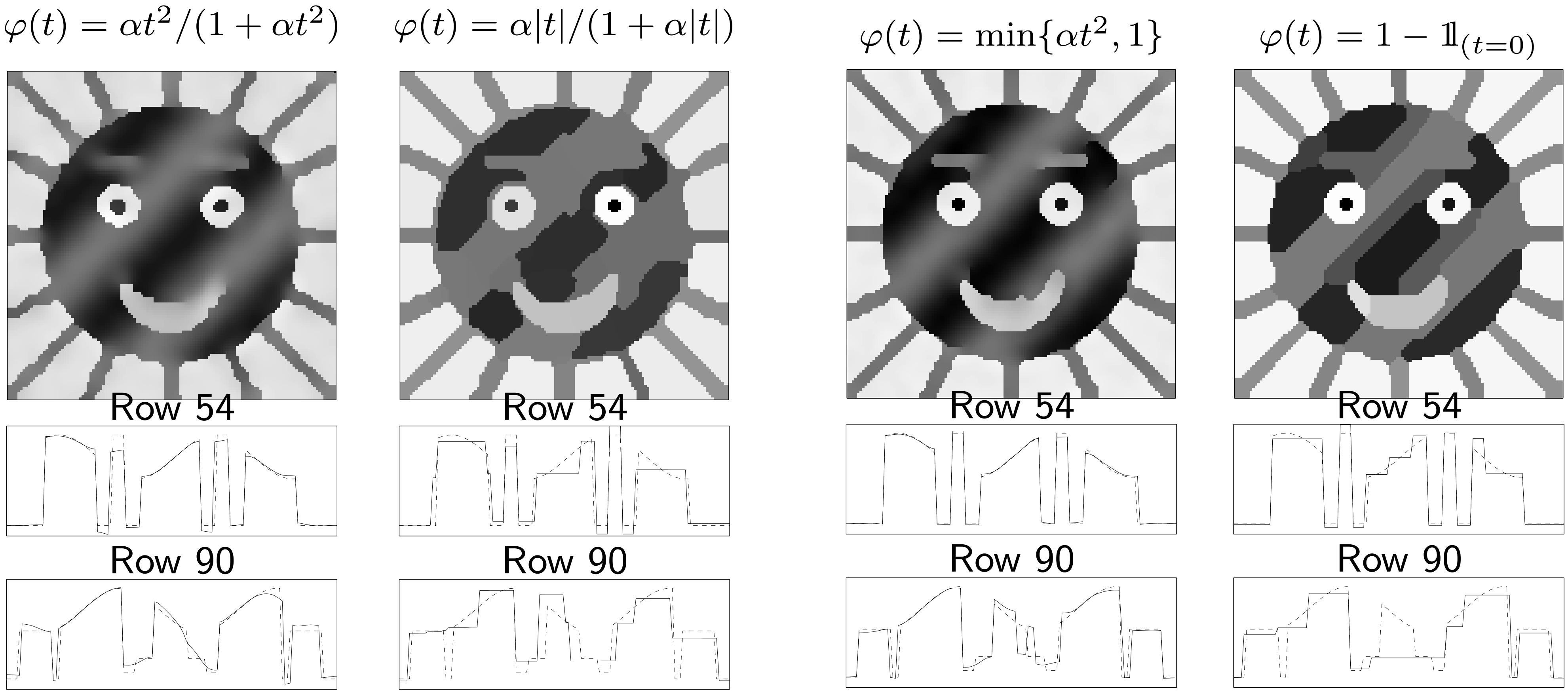
n is white Gaussian noise, SNR=20 dB

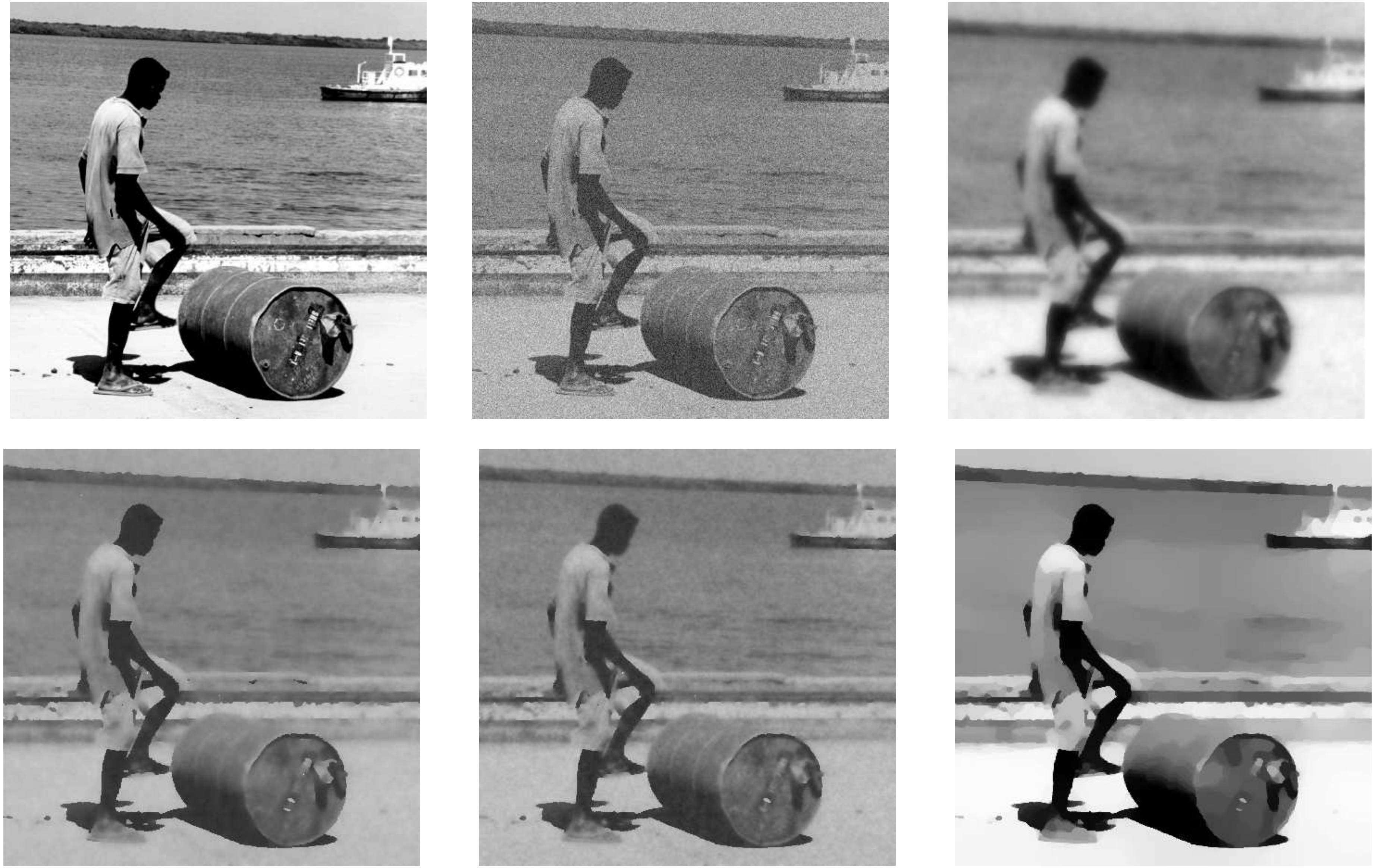
$\varphi(t) = |t|^\alpha$



$\varphi(t) = |t|$




$\varphi(t) = \alpha t^2/(1+\alpha t^2)$
$\varphi(t) = \alpha|t|/(1+\alpha|t|)$
$\varphi(t) = \min\{\alpha t^2, 1\}$
$\varphi(t) = 1 - \mathbb{1}_{(t=0)}$
Row 54
Row 90
Row 54
Row 90
Row 54
Row 90
Row 54
Row 90



7. Conclusions

- Minimizers relevant to non-convex regularization are stable and do involve smoothing
- Non-convex regularization enhances edges
- *Differences either smaller than a small threshold, or larger than a large threshold*
- *Each local minimum corresponds to an edge configuration*
- *Enhanced stair-casing when* φ *non-smooth at zero*
- Basically different from convex edge-preserving regularization
- Mathematical properties on the relation between the shape of a cost-function and the features of its minimizers are useful...

Papers available at `http://www.cmla.ens-cachan.fr/ nikolova/`