Mathematics in Imaging

Inverse modelling to solve imaging tasks using optimization

Mila Nikolova

CMLA, ENS Cachan, CNRS, UniverSud
61 Av. President Wilson, F-94230 Cachan, France

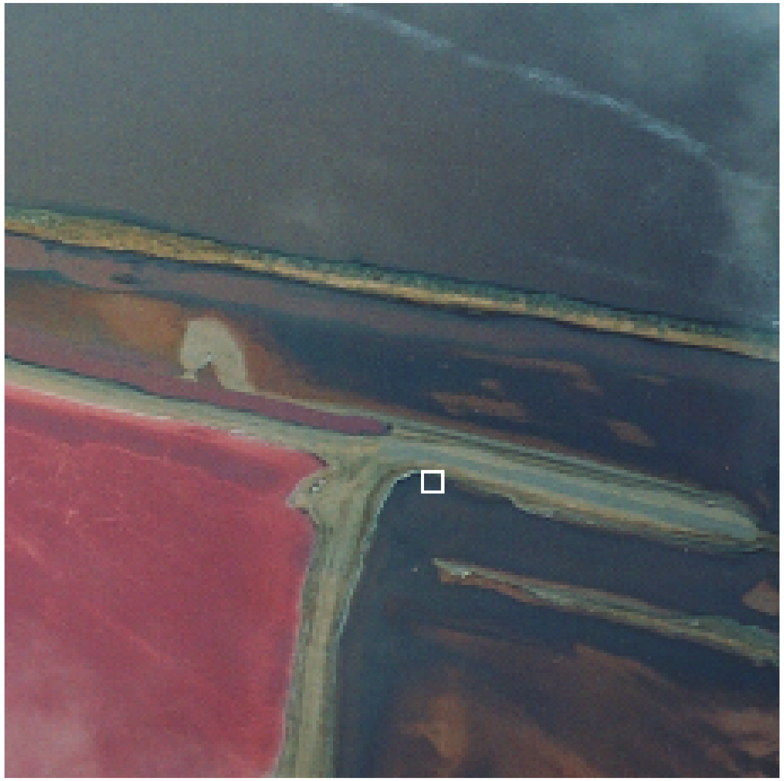
nikolova@cmla.ens-cachan.fr

Annual Meeting of the German Mathematical Society (DMV)

Universität des Saarlandes

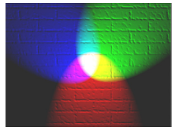
September 2012

Imaging science is a young field that arose with the "digital revolution"



Digital image $=$ 2D matrix with 3D entries:

$[\mathbf{RGB}]$



Entries are in a finite set of integers

8-bit images $\{0, \cdots, 255\}$

9-bit images $\{0, \cdots, 511\}$

Computers process only digital images

$$\begin{bmatrix} 48 & 54 & 48 & 51 & 59 \\ 47 & 46 & 49 & 50 & 52 \\ 48 & 50 & 48 & 48 & 51 \\ 56 & 51 & 54 & 52 & 50 \\ 57 & 58 & 54 & 51 & 48 \end{bmatrix} \begin{bmatrix} 74 & 79 & 74 & 71 & 77 \\ 76 & 71 & 77 & 75 & 77 \\ 81 & 81 & 80 & 81 & 79 \\ 90 & 83 & 85 & 87 & 80 \\ 89 & 89 & 88 & 84 & 80 \end{bmatrix} \begin{bmatrix} 64 & 70 & 63 & 61 & 70 \\ 64 & 64 & 66 & 61 & 64 \\ 66 & 65 & 63 & 64 & 64 \\ 76 & 71 & 70 & 71 & 63 \\ 75 & 72 & 66 & 68 & 70 \end{bmatrix}$$

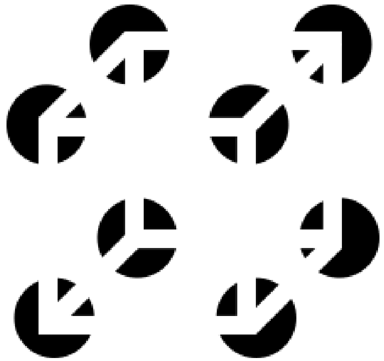
Natural images

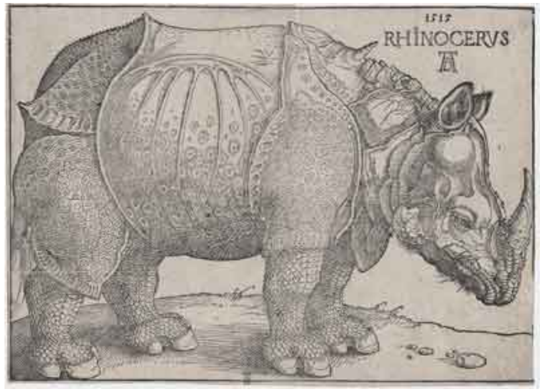
- Finite number of cells in the primary visual cortex
- Each cell recognizes a specific geometric shape or color data (D. Hubel, T. Wiesel)
- The whole image is produced in another part of our brain

> *"... our perceptions or ideas arise from an active critical principle."* *J.-J. Rousseau*

Gestalt theory of visual perception (since M. Wertheimer, 1923)

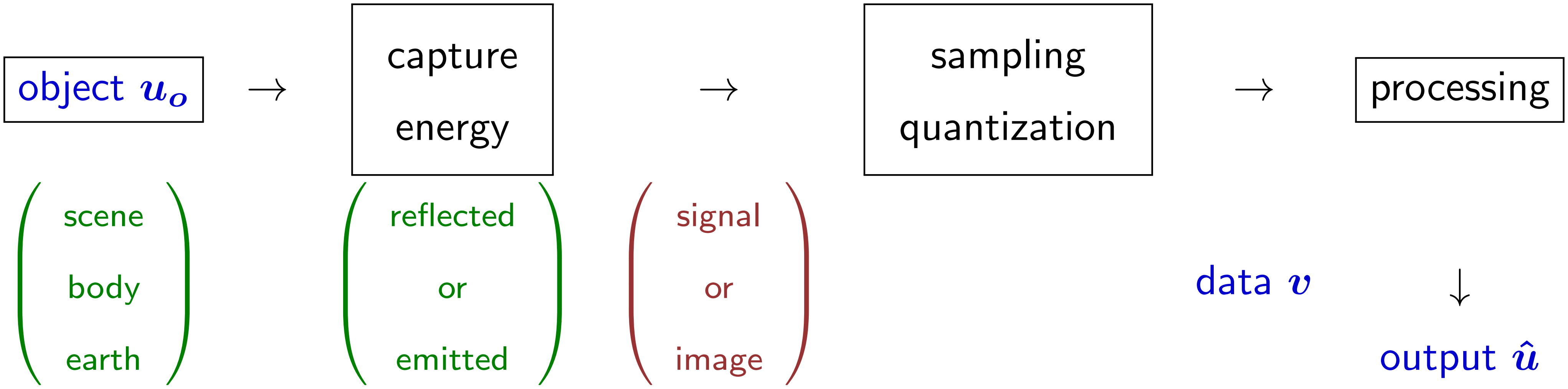
Is there a cube?





The output of imaging devices must satisfy perception (simplifications are enabled)

Objective criteria for image quality is an open question



Mathematical model: $\boldsymbol{v} = \text{Transform}(\boldsymbol{u_o}) \bullet (\text{Perturbations})$

Some transforms: loss of pixels, blur, FT, Radon T., frame T. $(\cdots)$

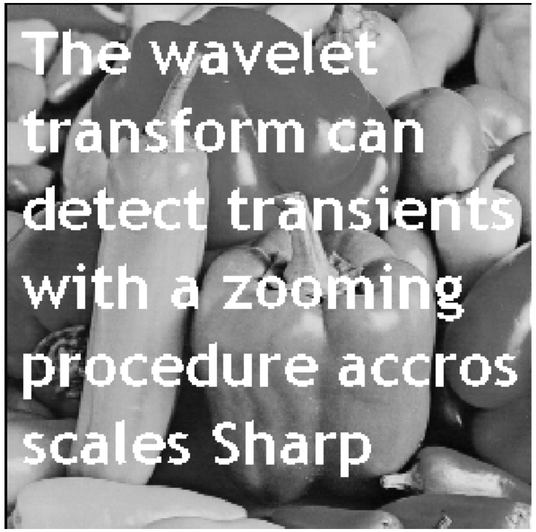
Processing tasks:
$$\begin{cases} \hat{\boldsymbol{u}} = \text{compress/code}(\boldsymbol{v}), \quad \text{transmit } \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{u}} = \text{recover}(\boldsymbol{u_o}) \\ \hat{\boldsymbol{u}} = \text{objects of interest}(\boldsymbol{u_o}) \end{cases} \quad (\cdots)$$

Mathematical tools: PDEs, Statistics, Functional anal., Matrix anal., Measure theory $(\cdots)$

Editing

↓

[P. Pérez, M. Gangnet and A. Blake, 2004]



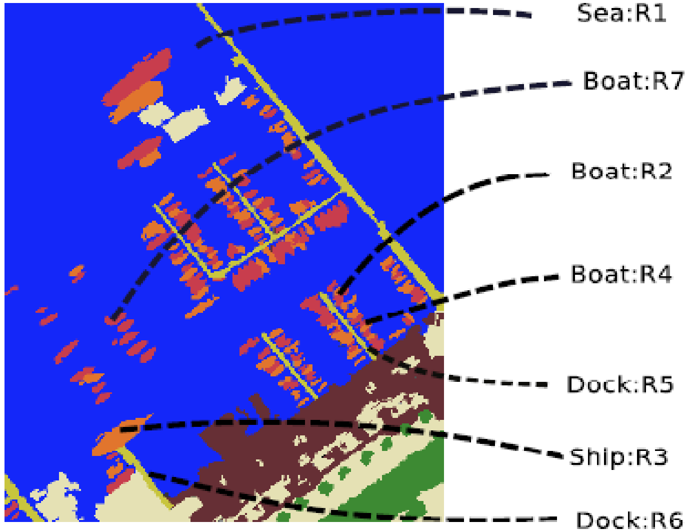
Inpainting

↓

[R. Chan, G. Steidl and S. Setzer, 2008]

Interpretation

↓



[I. Bloch, 2112]

Denoising

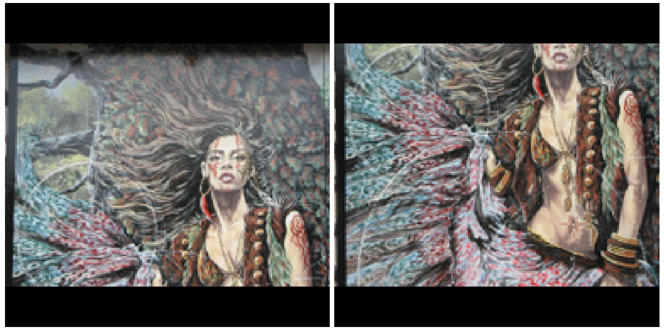
↓

[M. Lebrun, A. Buades and J.-M. Morel, 2112]

Line detection

↓

[R.G. von Gioi, J. Jakubowicz, J.-M. Morel, G. Randall, 12]



Registration

↓

[L. Moisan, P. Moulon and P. Monasse, 2112]

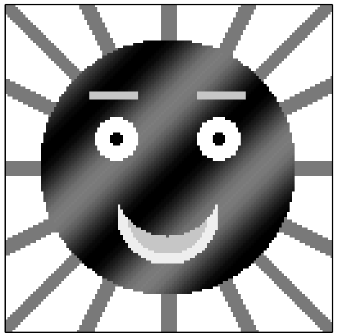
Image/signal processing tasks often require to solve **ill-posed inverse problems**

Out-of-focus picture: $v = a * u_o + \text{noise} = Au_o + \text{noise}$

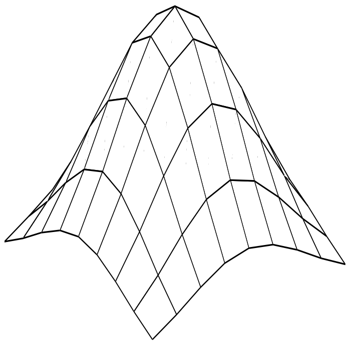
A is ill-conditioned $\equiv$ (nearly) noninvertible

Least-squares solution: $\hat{u} = \arg\min_{u} \left\{ \|Au - v\|^2 \right\}$

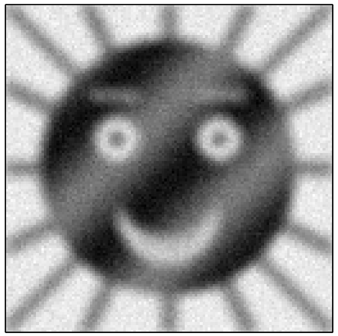
Tikhonov regularization: $\hat{u} \doteq \arg\min_{u} \left\{ \|Au - v\|^2 + \beta \sum_i \|G_i u\|^2 \right\}$ for $\{G_i\} \approx \nabla$, $\beta > 0$



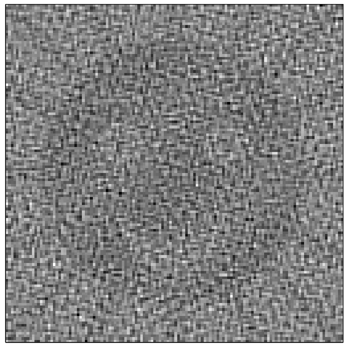
Original u_o



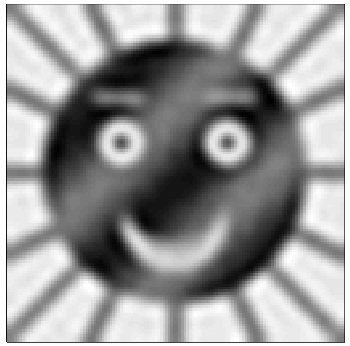
Blur a



Data v



$\hat{u}$: Least-squares



$\hat{u}$: Tikhonov

1. Energy minimization methods

u_o (unknown) $\quad v$ (data) $=$ Transform(u_o)$\bullet$ (Perturbations)

solution $\hat{u}$ $\nearrow$ close to data production model $\Psi(u,v)$ (data-fidelity)

$\searrow$ coherent with priors and desiderata $\Phi(u)$ (prior)

Combining models:

$$\hat{u} = \arg\min_{u\in\Omega} \mathcal{F}_v(u) \qquad (\mathcal{P})$$

$$\mathcal{F}_v(u) = \Psi(u,v) + \beta\Phi(u), \qquad \beta > 0$$

How to choose $(\mathcal{P})$ to get a good $\hat{u}$?

Applications: Denoising, Segmentation, Deblurring, Tomography, Seismic imaging, Zoom, Superresolution, Compression, Learning, Motion estimation, Pattern recognition $(\cdots)$

the $m \times n$ image u is stored in a $p = mn$-length vector, $u \in \mathbb{R}^p$, data $v \in \mathbb{R}^q$

$\boldsymbol{\Psi}$—usually models the production of data $\boldsymbol{v}$ $\Rightarrow$ $\Psi = -\log\big(\text{ Likelihood } (\boldsymbol{v}|\boldsymbol{u})\ \big)$

$v = Au_o + n$ for n white Gaussian noise $\Rightarrow$ $\boxed{\boldsymbol{\Psi(u,v) \propto \|Au - v\|_2^2}}$

Regularization [Tikhonov, Arsenin 77]: $\boldsymbol{\mathcal{F}_v(u) = \|Au - v\|^2 + \beta\|Gu\|^2}$, $G = I$ or $G \approx \nabla$

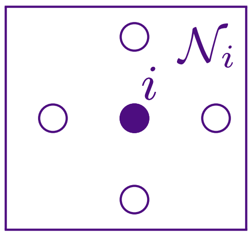
Focus on edges and contours

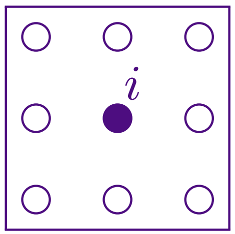
Statistical framework

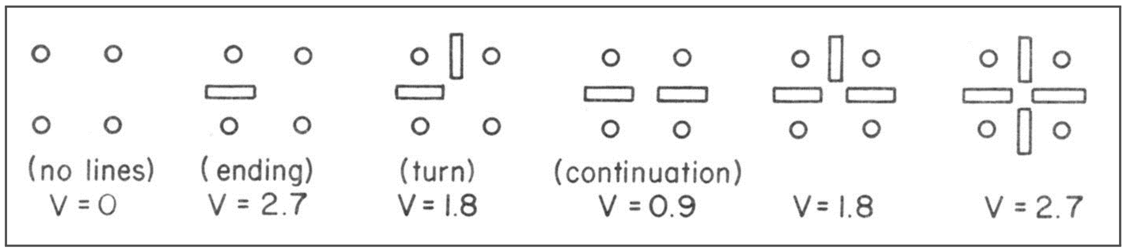
Line process in Markov random field priors [Geman, Geman 84]: $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\ell}}) = \arg\min_{u,\ell} \mathcal{F}_v(u,\ell)$

$$\boldsymbol{\mathcal{F}_v(u,\ell) = \Psi(u,v) + \beta \sum_i \Big(\sum_{j \in \mathcal{N}_i} \varphi(u[i] - u[j])(1 - \ell_{i,j}) + \sum_{(k,n) \in \mathcal{N}_{i,j}} \mathbf{V}(\ell_{i,j}, \ell_{k,n}) \Big)}$$

$[\ell_{i,j} = 0 \;\Leftrightarrow\;$ no edge$]$, $[\ell_{i,j} = 1 \;\Leftrightarrow\;$ edge between i and $j]$, $\varphi(t) = 1$







Computation: stochastic relaxation and annealing (global convergence with high probability)

PDE's framework

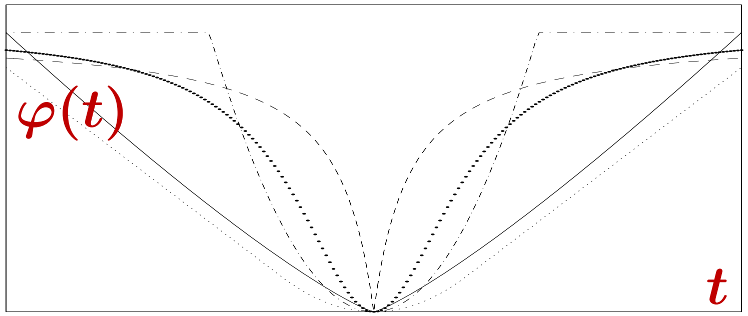
$\Phi(\mathbf{u})$

M.-S. functional [Mumford, Shah 89]: $\mathcal{F}_v(u, L) = \int_\Omega (u-v)^2 dx + \beta \left(\int_{\Omega \setminus L} \|\nabla u\|^2 dx + \alpha |L| \right)$

discrete version: $\Phi(u) = \sum_i \varphi(\|G_i u\|)$, $\quad \varphi(t) = \min\{t^2, \alpha\}$, $\quad \{G_i\} \approx \nabla$

Total Variation (TV) [Rudin, Osher, Fatemi 92]: $\mathcal{F}_v(u) = \|u - v\|_2^2 + \beta\, \mathrm{TV}(u)$

$$\mathrm{TV}(u) = \int \|\nabla u\|_2\, dx \approx \sum_i \|G_i u\|_2$$



various functions φ to define Φ

φ is edge-preserving if $\lim_{t \to \infty} \frac{\varphi'(t)}{t} = 0$

[Charbonnier, Blanc-Féraud, Aubert, Barlaud 97]

Minimizer approach

ℓ_1- Data fidelity [Nikolova 02]: $\mathcal{F}_v(u) = \|Au - v\|_1 + \beta \Phi(u)$

$L_1 - \mathbf{TV}$ model [T. Chan, Esedoglu 05]: $\mathcal{F}_v(u) = \|u - v\|_1 + \beta\, \mathrm{TV}(u)$

CPU time ! Computers ↑↑

Minimizer approach [Nikolova 96] :

Analyze the mutual relation between the shape of $(\mathcal{P})$
and the main features exhibited by its solutions $\hat{u}$

Exhibit the main features of the (local) minimizers $\hat{u}$ of $\mathcal{F}_v$ as a function of the shape of $\mathcal{F}_v$ and the data v

(a "chicken and egg" problem?)

Strong results, control on the solutions

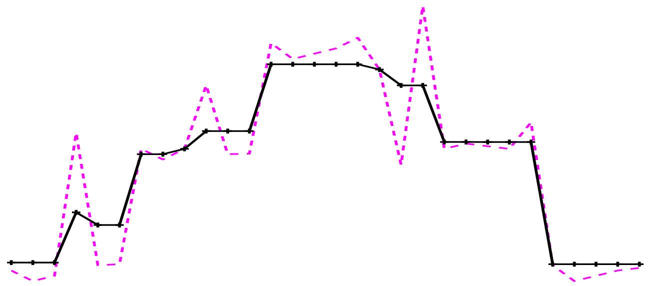
Rigorous tools for modelling

Goal:

Construct $\mathcal{F}_v$ so that the properties of $\hat{u}$ match the models for the data and the unknown

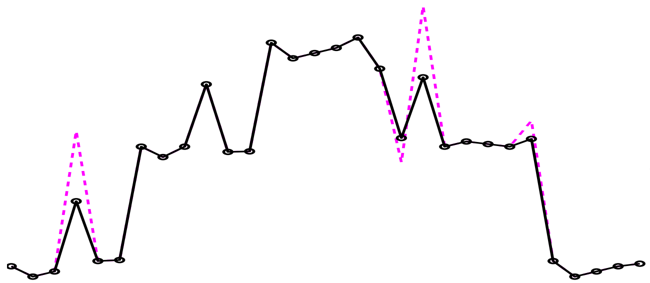
"There is nothing quite as practical as a good theory." *Kurt Lewin*

Illustration: the role of the smoothness of $\mathcal{F}_v$



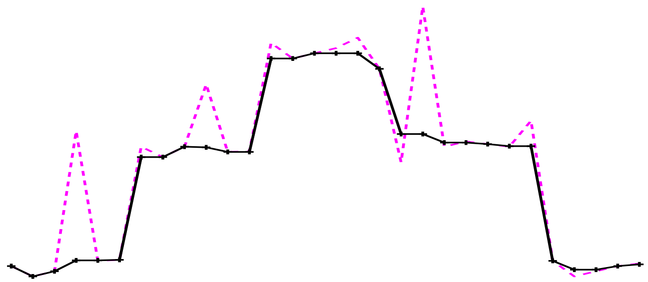
STAIR-CASING

$$\mathcal{F}_v(u) = \underbrace{\sum_{i=1}^{p}(u_i - v_i)^2}_{\textbf{smooth}} + \beta \underbrace{\sum_{i=1}^{p-1}|u_i - u_{i+1}|}_{\textbf{non-smooth}}$$



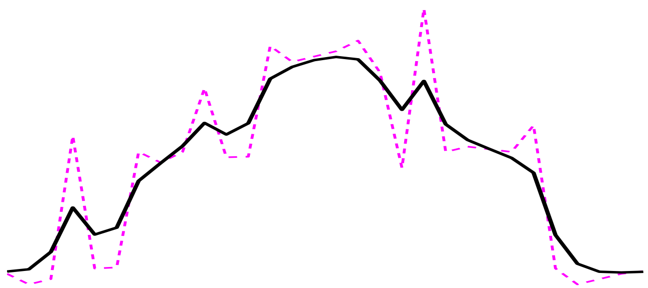
EXACT DATA-FIT

$$\mathcal{F}_v(u) = \underbrace{\sum_{i=1}^{p}|u_i - v_i|}_{\textbf{non-smooth}} + \beta \underbrace{\sum_{i=1}^{p-1}(u_i - u_{i+1})^2}_{\textbf{smooth}}$$



BOTH EFFECTS

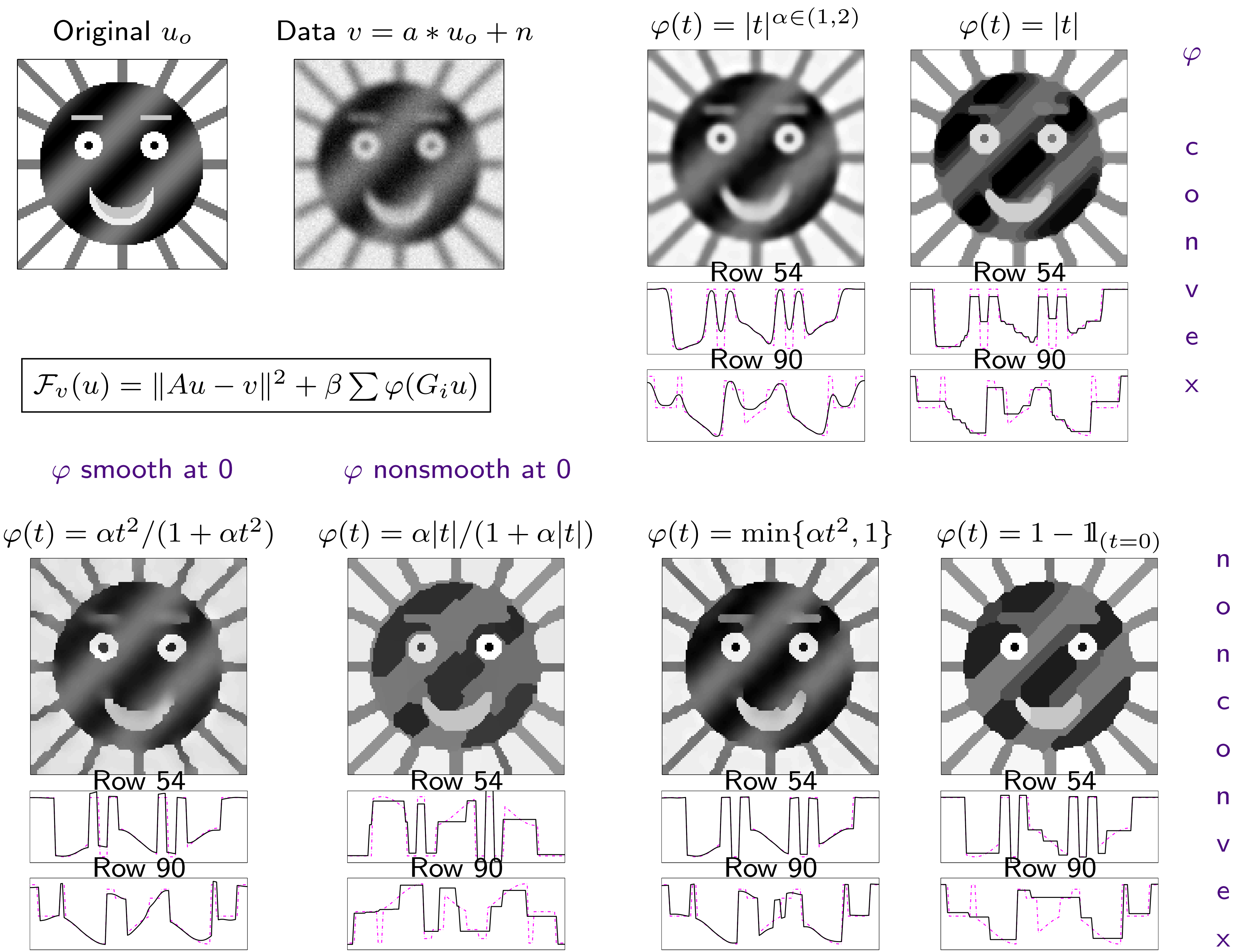
$$\mathcal{F}_v(u) = \underbrace{\sum_{i=1}^{p}|u_i - v_i|}_{\textbf{non-smooth}} + \beta \underbrace{\sum_{i=1}^{p-1}|u_i - u_{i+1}|}_{\textbf{non-smooth}}$$



$$\mathcal{F}_v(u) = \underbrace{\sum_{i=1}^{p}(u_i - v_i)^2}_{\textbf{smooth}} + \beta \underbrace{\sum_{i=1}^{p-1}(u_i - u_{i+1})^2}_{\textbf{smooth}}$$

Data (— — —), Minimizer (—)

We shall explain why



Original u_o
Data $v = a * u_o + n$
$\varphi(t) = |t|^{\alpha \in (1,2)}$
$\varphi(t) = |t|$
Row 54
Row 90
Row 54
Row 90
$\mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum \varphi(G_i u)$
φ convex
φ smooth at 0
φ nonsmooth at 0
$\varphi(t) = \alpha t^2/(1 + \alpha t^2)$
$\varphi(t) = \alpha|t|/(1 + \alpha|t|)$
$\varphi(t) = \min\{\alpha t^2, 1\}$
$\varphi(t) = 1 - 1\!\!1_{(t=0)}$
Row 54
Row 90
Row 54
Row 90
Row 54
Row 90
Row 54
Row 90
nonconvex

2 Regularity results

$$\begin{aligned} \mathcal{F}_v(u) &= \|Au - v\|_2^2 + \beta\Phi(u) \\ \Phi(u) &= \sum_i \varphi(\|G_i u\|_2) \end{aligned}$$

$u \in \mathbb{R}^p$

$v \in \mathbb{R}^q$

$$\begin{cases} \varphi : \mathbb{R}_+ \to \mathbb{R} \\ \varphi \text{ incresing, continuous} \\ \varphi(t) > \varphi(0), \quad \forall t > 0 \end{cases}$$

$\{G_i\}$ linear operators $\mathbb{R}^p \to \mathbb{R}^s$

If $\varphi'(0^+) > 0 \quad \Rightarrow \quad \Phi$ is nonsmooth on $\bigcup_i \{u : G_i u = 0\}$

Systematically: $\ker A \cap \ker G = \{0\}$

$$\left(G = \begin{bmatrix} G_1 \\ G_2 \\ \cdots \end{bmatrix}, \text{ in general } \ker G \supsetneq \{0\} \right)$$

Definition: $\mathcal{U} : O \to \mathbb{R}^p$, $O \subset \mathbb{R}^q$ open, is a (local) minimizer function for

$\mathcal{F}_O \doteq \{\mathcal{F}_v \ : \ v \in O\}$ if $\mathcal{F}_v$ has a strict (local) minimum at $\mathcal{U}(v)$, $\forall\, v \in O$

$\mathcal{F}_v$ nonconvex $\Rightarrow$ there may be many local minima

2.1 Stability of the minimizers of $\mathcal{F}_v$ *[Durand & Nikolova 06]*

Assumptions: φ is piecewise $\mathcal{C}^{m\geqslant 2}$, edge-preserving, non-convex and $\mathrm{rank}(A) = p$

LOCAL MINIMIZERS

`(knowing local minimizers is important)`

There is a closed $N \subset \mathbb{R}^q$ with Lebesgue measure $\mathbb{L}^q(\boldsymbol{N}) = \boldsymbol{0}$ such that $\forall \boldsymbol{v} \in \mathbb{R}^q \setminus \boldsymbol{N}$, every (local) minimizer $\hat{\boldsymbol{u}}$ of $\boldsymbol{\mathcal{F}_v}$ is given by $\hat{\boldsymbol{u}} = \boldsymbol{\mathcal{U}}(v)$ where $\boldsymbol{\mathcal{U}}$ is a $\mathcal{C}^{m-1}$ (local) minimizer function.

GLOBAL MINIMIZERS

- $\exists\ \hat{\boldsymbol{N}} \subset \mathbb{R}^q$ with $\mathbb{L}^q(\hat{\boldsymbol{N}}) = \boldsymbol{0}$ and $\mathrm{Int}(\mathbb{R}^q \setminus \hat{N})$ dense in $\mathbb{R}^q$ such that $\forall \boldsymbol{v} \in \mathbb{R}^q \setminus \hat{\boldsymbol{N}}$, $\boldsymbol{\mathcal{F}_v}$ has a unique global minimizer.
- There is an open subset of $\mathbb{R}^q \setminus \hat{N}$, dense in $\mathbb{R}^q$, where the global minimizer function $\hat{\mathcal{U}}$ is $\mathcal{C}^{m-1}$-continuous.

2.2 Nonasymptotic bounds on minimizers

[Nikolova 07]

Classical bounds for $\beta \searrow 0$ or $\beta \nearrow \infty$

Assumption: φ is piecewise $\mathcal{C}^1$

- φ is strictly increasing **or** $\mathrm{rank}(A) = p$

$$\hat{u} \text{ is a (local) minimizer of } \mathcal{F}_v \quad \Rightarrow \quad \|A\hat{u}\| \leqslant \|v\|$$

- $\|\varphi'\|_\infty < \infty$ (φ is edge-preserving) and $\mathrm{rank}(A) = q \leqslant p$

$$\hat{u} \text{ is a (local) minimizer of } \mathcal{F}_v \quad \Rightarrow \quad \|v - A\hat{u}\|_\infty \leqslant \frac{\beta}{2} \, \|\varphi'\|_\infty \, \|(AA^*)^{-1}A\|_\infty \, \|G\|_1$$

$\|\varphi'\|_\infty = 1$, $A = \mathrm{Id}$ and G – 1^{st} order:
$$\begin{cases} \text{signal } (\|G\|_1 = 2) & \Rightarrow \quad \|v - \hat{u}\|_\infty \leqslant \beta \\ \text{image } (\|G\|_1 = 4) & \Rightarrow \quad \|v - \hat{u}\|_\infty \leqslant 2\beta \end{cases}$$

Surprising?

3 Minimizers under Non-Smooth Regularization

[Nikolova 97,01,04]

$$\mathcal{F}_v(u) = \Psi(u,v) + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|), \quad \Psi \in \mathcal{C}^{m \geqslant 2}, \; \varphi \in \mathcal{C}^m(\mathbb{R}_+^*), \; 0 < \varphi'(0^+) \leqslant \infty$$

$$\varphi(t) \;\Big\|\; t^\alpha,\; \alpha \in (0,1) \;\Big|\; \frac{\alpha\, t}{\alpha\, t + 1} \;\Big|\; \ln(\alpha t + 1) \;\Big|\; 0 \text{ if } t = 0,\; 1 \text{ if } t \neq 0 \;\Big|\; (\cdots) \;, \quad \alpha > 0$$

Let $\hat{u}$ be a (local) minimizer of $\mathcal{F}_v$. Set $\hat{h} \doteq \{i : G_i \hat{u} = 0\}$

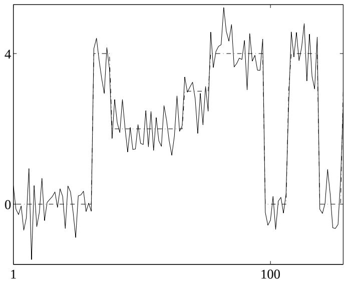
Then $\exists$ $O \subset \mathbb{R}^q$ **open,** $\exists\, \mathcal{U} \in \mathcal{C}^{m-1}$ (local) minimizer function so that

$$v' \in O, \quad \hat{u}' = \mathcal{U}(v') \quad \Rightarrow \quad G_i \hat{u}' = 0, \quad \forall i \in \hat{h}$$

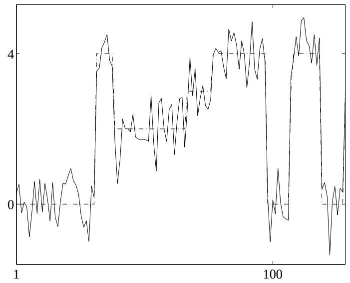
Data v yield (local) minimizers $\hat{u}$ of $\mathcal{F}_v$ such that $G_i \hat{u} = 0$ for a set of indexes $\hat{h}$

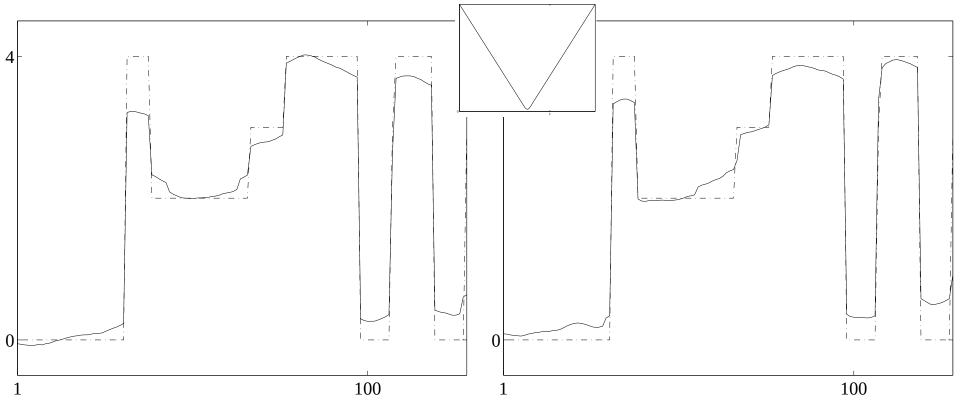
$G_i = \nabla_i \;\Rightarrow\; \hat{u}[i] = \hat{u}[j]$ for many neighbors (i,j) (the “stair-casing” effect)

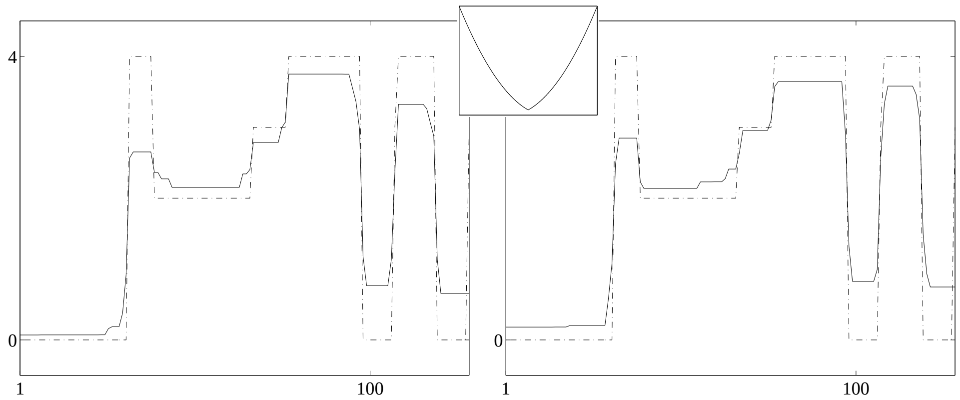
$G_i u = u[i] \;\Rightarrow$ many samples $\hat{u}[i] = 0$ – highly used in Compressed Sensing



$$\mathcal{F}_v(u) = \|u - v\|^2 + \beta \sum \varphi(u[i] - u[i-1])$$

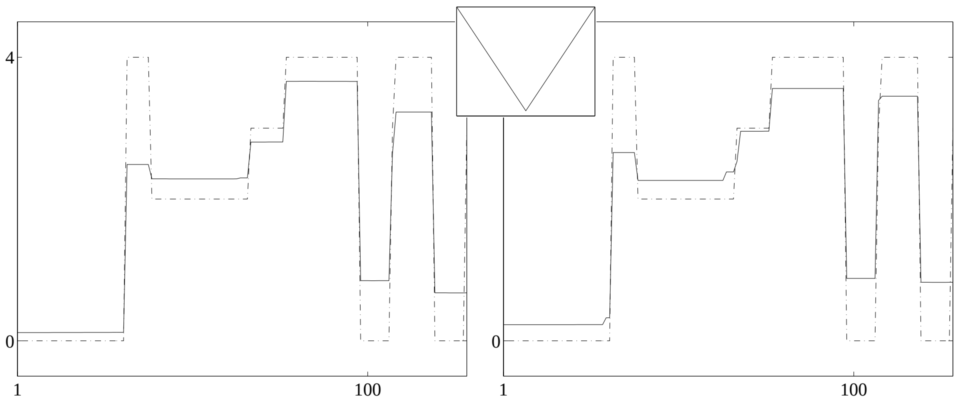


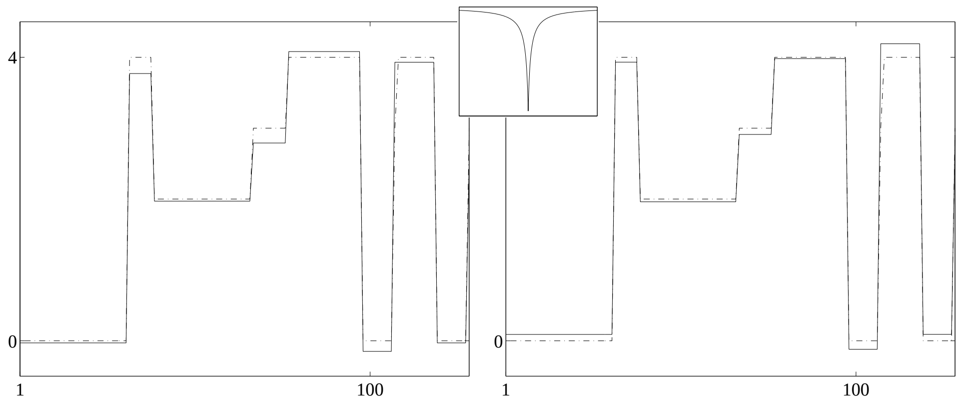




$\varphi(t) = \sqrt{\alpha + t^2}, \quad \varphi'(0) = 0$ **(smooth at 0)**

$\varphi(t) = (t + \alpha \mathrm{sign}(t))^2, \quad \varphi'(0^+) = 2\alpha$

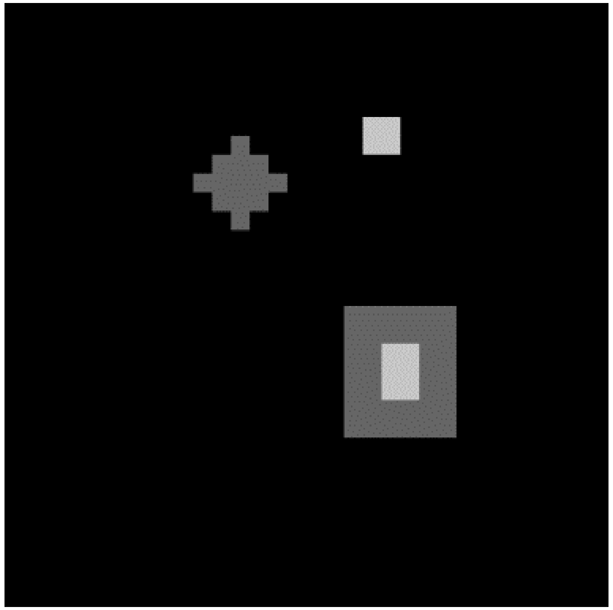




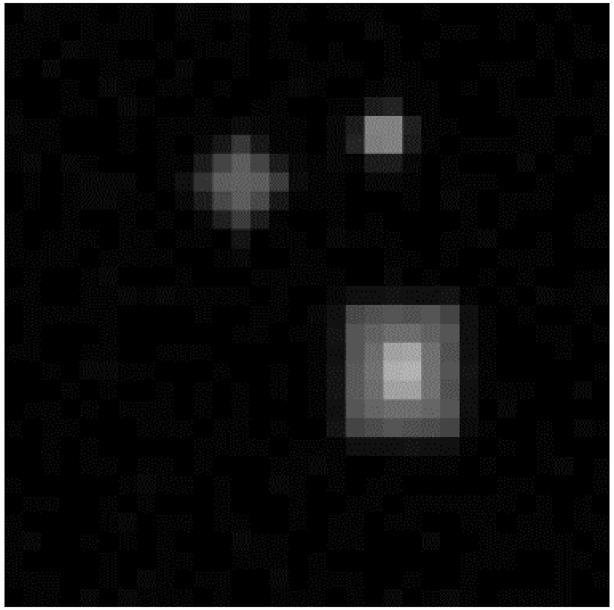
$\varphi(t) = |t|, \quad \varphi'(0^+) = 1$

$\varphi(t) = \alpha|t|/(1 + \alpha|t|), \quad \varphi'(0^+) = \alpha$

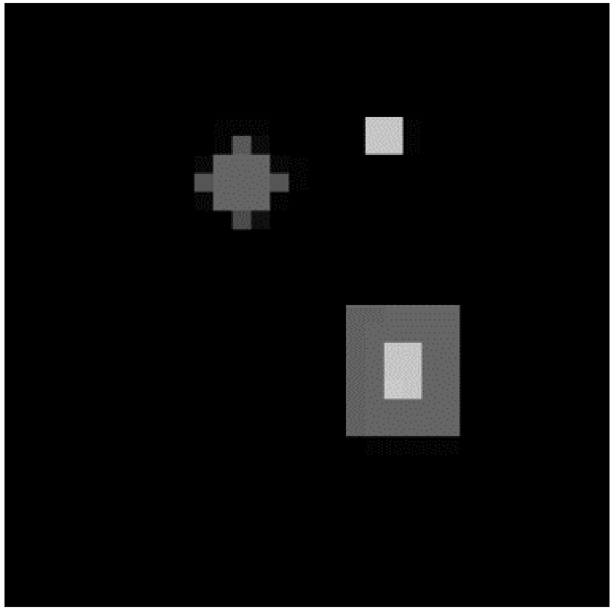
TV energy: $\mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|)$ for $\varphi(t) = t$ and G_i discrete gradient at pixel i



Original



Data



Restored: TV energy

D. C. Dobson and F. Santosa, "Recovery of blocky images from noisy and blurred data", SIAM J. Appl. Math., 56 (1996), pp. 1181-1199.

4 Minimizers relevant to non-smooth data-fidelity

4.1 General case

[Nikolova 01,02]

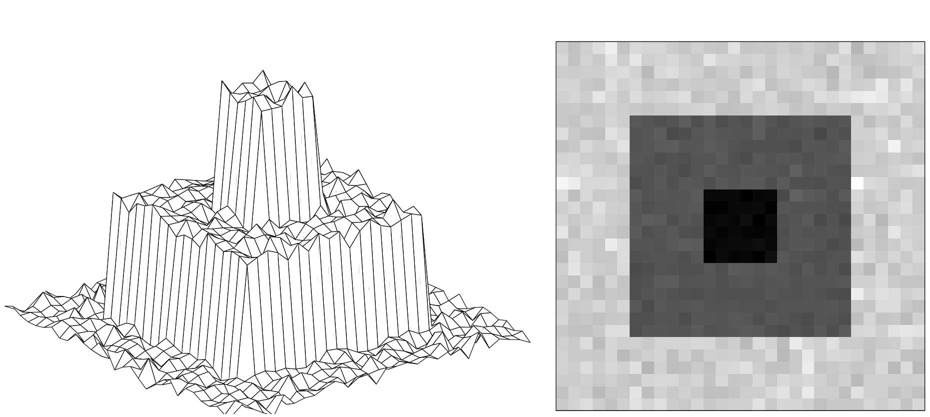
$$\mathcal{F}_v(u)=\sum_i \psi(|a_i u - v[i]|) + \beta\Phi(u), \qquad \Phi\in\mathcal{C}^m,\ \psi\in\mathcal{C}^m(\mathbb{R}_+^*),\quad \psi'(0^+)>0$$

Let $\hat{u}$ be a (local) minimizer of $\mathcal{F}_v$. Set $\hat{h}=\{i : a_i\hat{u}=v[i]\}$.
Then $\exists\ O\subset\mathbb{R}^q$ open, $\exists\ \mathcal{U}\in\mathcal{C}^{m-1}$ (local) minimizer function so that

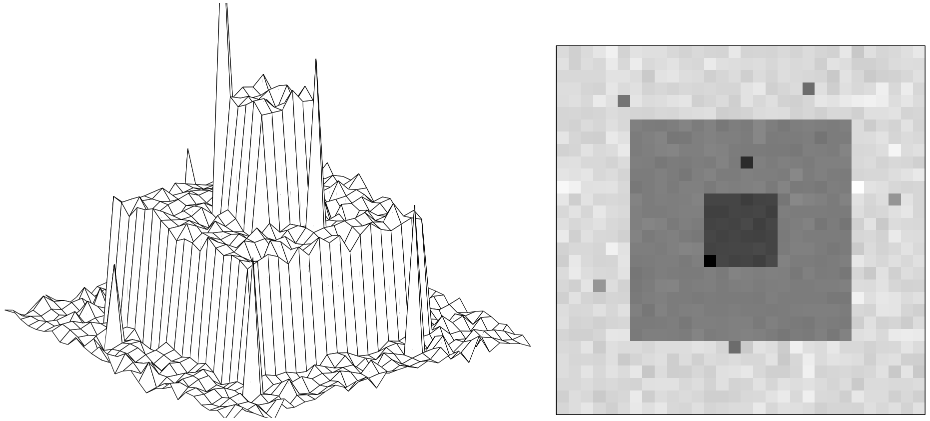
$$v'\in O,\quad \hat{u}'=\mathcal{U}(v') \quad\Rightarrow\quad \begin{cases} a_i\hat{u}'=v[i], & i\in\hat{h}\\ a_i\hat{u}'\neq v[i], & i\in\hat{h}^c\end{cases}$$

(Local) minimizers $\hat{u}$ of $\mathcal{F}_v$ achieve an **exact fit** to (noisy) data
$a_i\hat{u}=v[i]$ for a certain number of indexes i

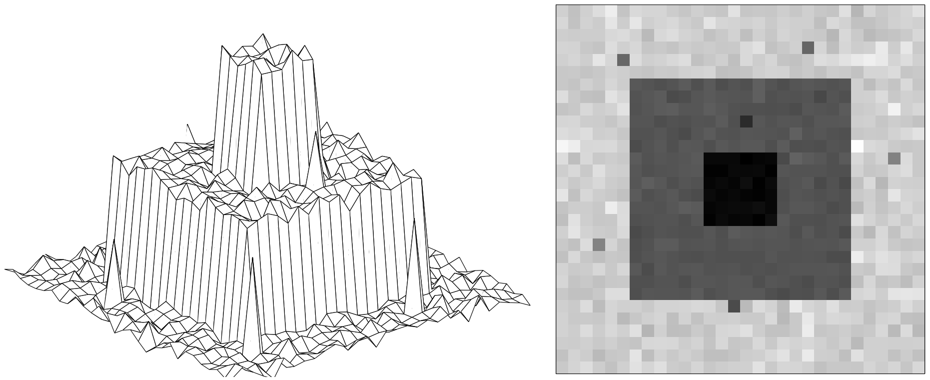
Property does not hold if $\mathcal{F}$ is smooth, except for $v\in N$ where N is closed and $\mathbb{L}^q(N)=0$.



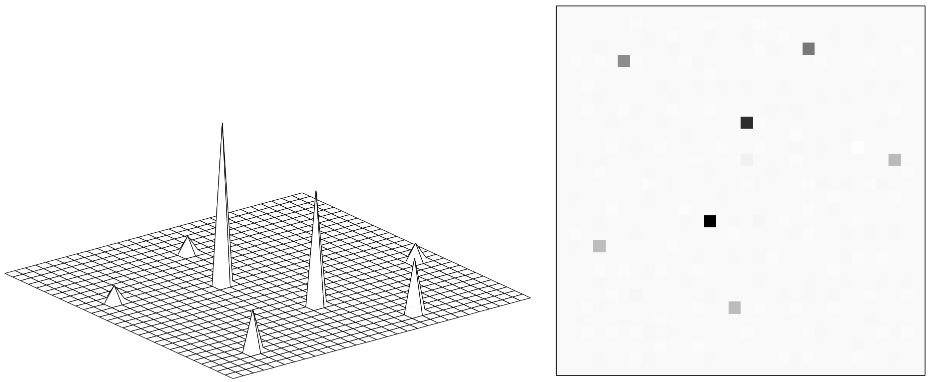
Original u_o



Data $v = u_o$+outliers

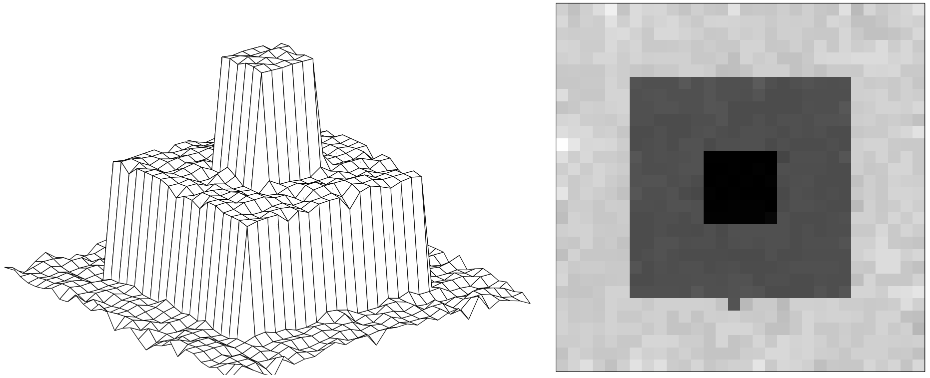


Restoration $\hat{u}$ for $\boldsymbol{\beta = 0.14}$

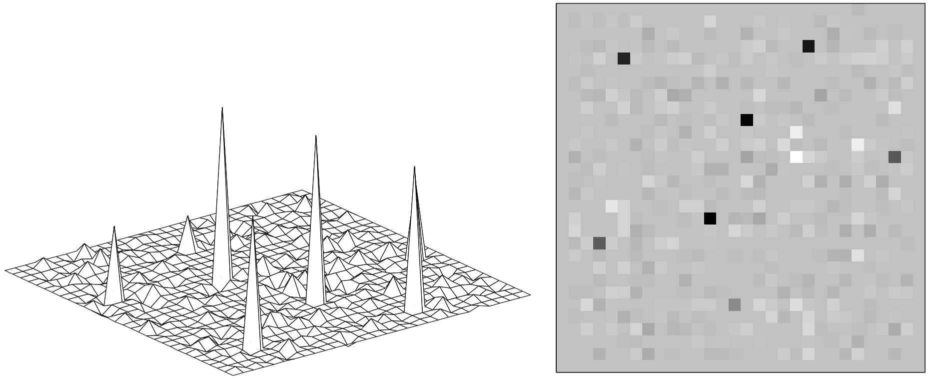


Residuals $v - \hat{u}$

$$\boldsymbol{\mathcal{F}_v(u) = \sum_i |u[i] - v[i]| + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]|^{1.1}}$$

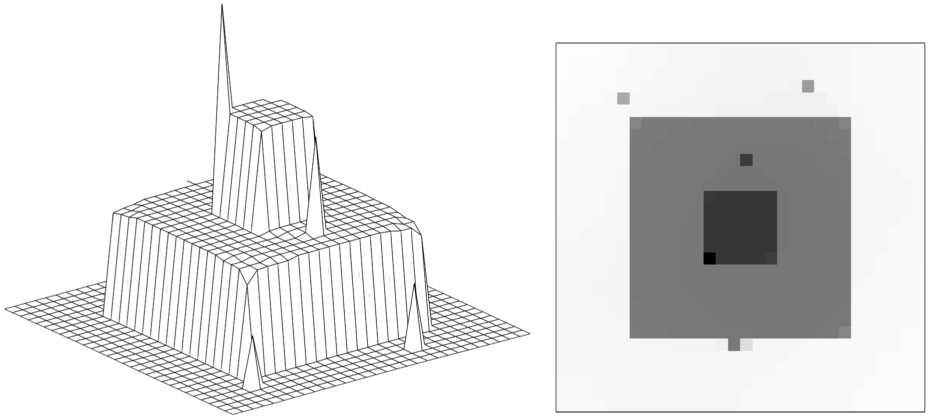


Restoration $\hat{u}$ for $\boldsymbol{\beta = 0.25}$

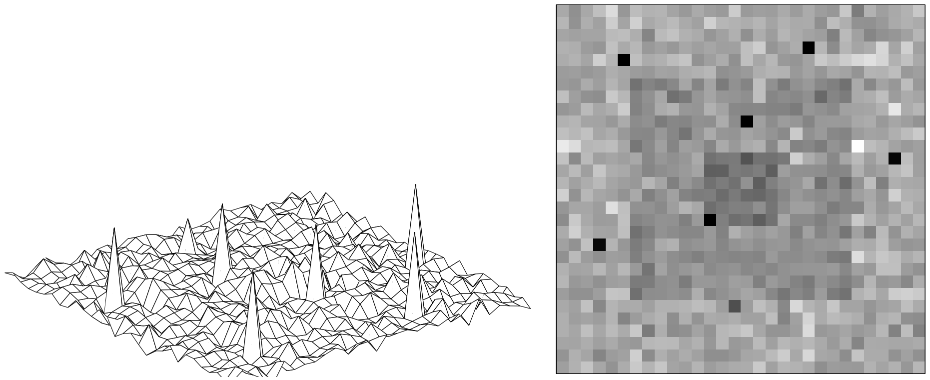


Residuals $v - \hat{u}$

$$\boldsymbol{\mathcal{F}_v(u) = \sum_i \big|u[i] - v[i]\big| + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]|^{1.1}}$$



Restoration $\hat{u}$ for $\beta = 0.2$

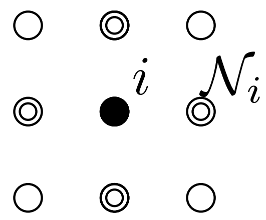


Residuals $v - \hat{u}$

TV-like energy: $\boldsymbol{\mathcal{F}_v(u) = \sum_i (u[i] - v[i])^2 + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]|}$

4.2 Detection and cleaning of outliers using ℓ_1 data-fidelity [Nikolova 04]

$$\boxed{\mathcal{F}_v(u) = \sum_{i=1}^{p} |u[i] - v[i]| + \frac{\beta}{2} \sum_{i=1}^{p} \sum_{j \in \mathcal{N}_i} \varphi(|u[i] - u[j]|)}$$



φ: smooth, convex, *edge-preserving*

Assumptions:
$$\begin{cases} \text{data } v \text{ contain uncorrupted samples } v[i] \\ v[i] \text{ is outlier if } \; |v[i] - v[j]| \gg 0, \;\; \forall j \in \mathcal{N}_i \end{cases}$$

$$v \in \mathbb{R}^p \quad \Rightarrow \quad \begin{array}{l} \hat{u} = \arg\min_u \mathcal{F}_v(u) \\ \hat{h} = \{i : \hat{u}[i] = v[i]\} \end{array} \qquad \begin{cases} v[i] \text{ is regular} & \text{if } i \in \hat{h} \\ v[i] \text{ is outlier} & \text{if } i \in \hat{h}^c \end{cases}$$

Outlier detector: $v \to \hat{h}^c(v) = \{i : \hat{u}[i] \neq v[i]\}$

Smoothing: $\hat{u}[i]$ for $i \in \hat{h}^c$ = estimate of the outlier

Justification based on the properties of $\hat{u}$

4.3 Recovery of frame coefficients using ℓ_1 data-fitting

[Durand, Nikolova 07]

- Data: $v = u_o +$ noise
- Frame coefficients: $y = Wv = Wu_o +$ noise
- Hard thresholding keeps relevant information if T small: $y_T[i] \doteq \begin{cases} 0 & \text{if } |y[i]| \leqslant T \\ y[i] & \text{if } |y[i]| > T \end{cases}$
- Hybrid energy methods—combine fitting to y_T with prior $\Phi(u)$

[Bobichon &Bijaoui 97, Coifman &Sowa 00, Durand &Froment 03...]

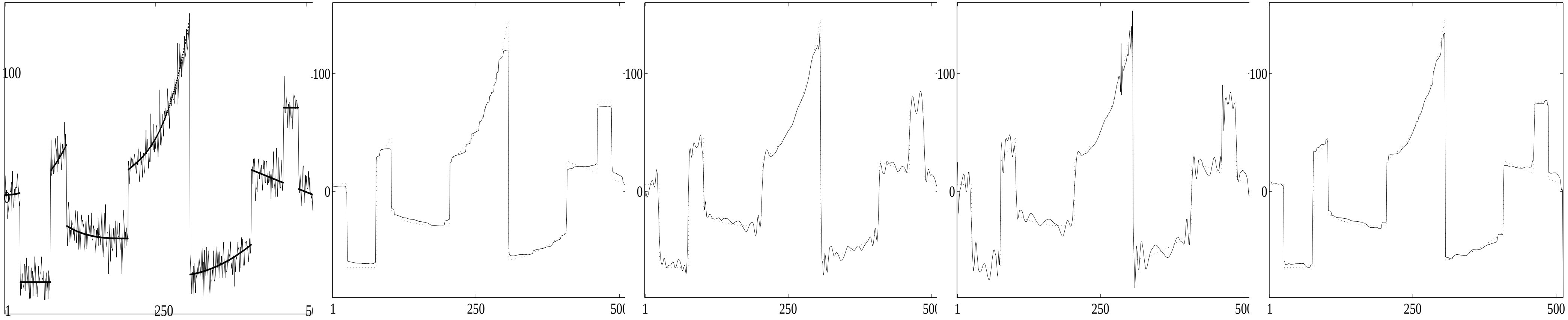
Desiderata: $\mathcal{F}_y$ convex and

Keep $\hat{x}[i] = y_T[i]$	Restore $\hat{x}[i] \neq y_T[i]$
significant coefs: $y[i] \approx (Wu_o)[i]$	outliers: $\lvert y[i]\rvert \gg \lvert (Wu_o)[i]\rvert$ (frame-shaped artifacts)
thresholded coefs: $(Wu_o)[i] \approx 0$	edge coefs: $\lvert (Wu_o)[i]\rvert > \lvert y_T[i]\rvert = 0$ ("Gibbs" oscillations)

Then:

minimize $\mathcal{F}_y(x) = \sum_i \lambda_i \big|(x - y_T)[i]\big| + \int_\Omega \varphi(|\nabla \widetilde{W} x|) \Rightarrow \hat{x}$

$\hat{u} = \widetilde{W}\hat{x}$ for $\widetilde{W}$ left inverse, φ edge-preserving



Original & data TV (Total Variation) Sure-Shrink T optimal Our method

4.4 Fast 2-stage restoration under impulse noise *[R.Chan, Nikolova et al. 04, 05, 08]*

1. Approximate the outlier-detection stage by rank-order filter

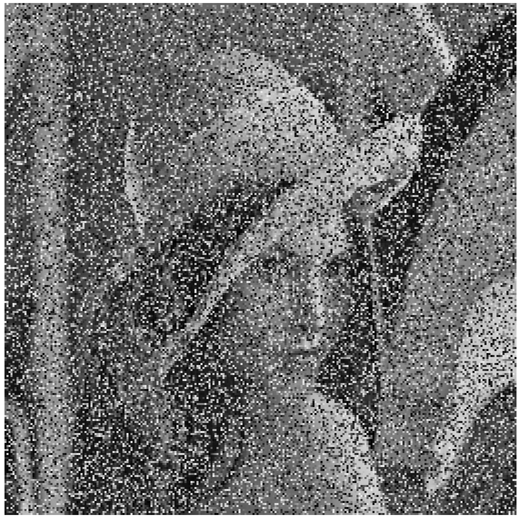
   (e.g. adaptive or center-weighted median)

   Corrupted pixels $\hat{h}^c = \{i : \hat{v}[i] \neq v[i]\}$ where $\hat{v}$=Rank-Order Filter (v)

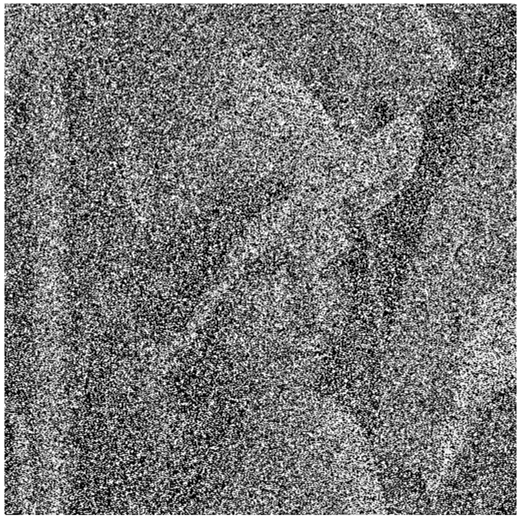
   $\Rightarrow$ improve speed and accuracy

2. Restore $\hat{u}$ (denoise, deblur) using an edge-preserving energy method

   subject to $a_i \hat{u} = v[i]$ for all $i \in \hat{h}$




50% RV noise | ACWMF | DPVM | Our method




70 %SP noise(6.7dB) | Adapt.med.(25.8dB) | Our method(29.3dB) | Original Lena

L. Bar, A. Brook, N. Sochen and N. Kiryati,
"Deblurring of Color Images Corrupted by Impulsive Noise",
IEEE Trans. on Image Processing, 2007

$$\mathcal{F}_v(u) = \|Au - v\|_1 + \beta\Phi(u)$$

blurred, noisy (r.-v.)

zoom - restored

4.5 One-step real-time dejittering of digital video

[Nikolova 09]

Image $\boldsymbol{u} \in \mathbb{R}^{r\times c}$, rows $\boldsymbol{u}_i$, pixels $\boldsymbol{u}_i(j)$

Data $\boldsymbol{v}_i(j) = \boldsymbol{u}_i(j + \boldsymbol{d}_i)$, $\boldsymbol{d}_i$ integer, $|\boldsymbol{d}_i| \leqslant \boldsymbol{M}$

Restore $\hat{\boldsymbol{u}} \equiv$ restore $\hat{\boldsymbol{d}}_i$, $\boldsymbol{1 \leqslant i \leqslant r}$

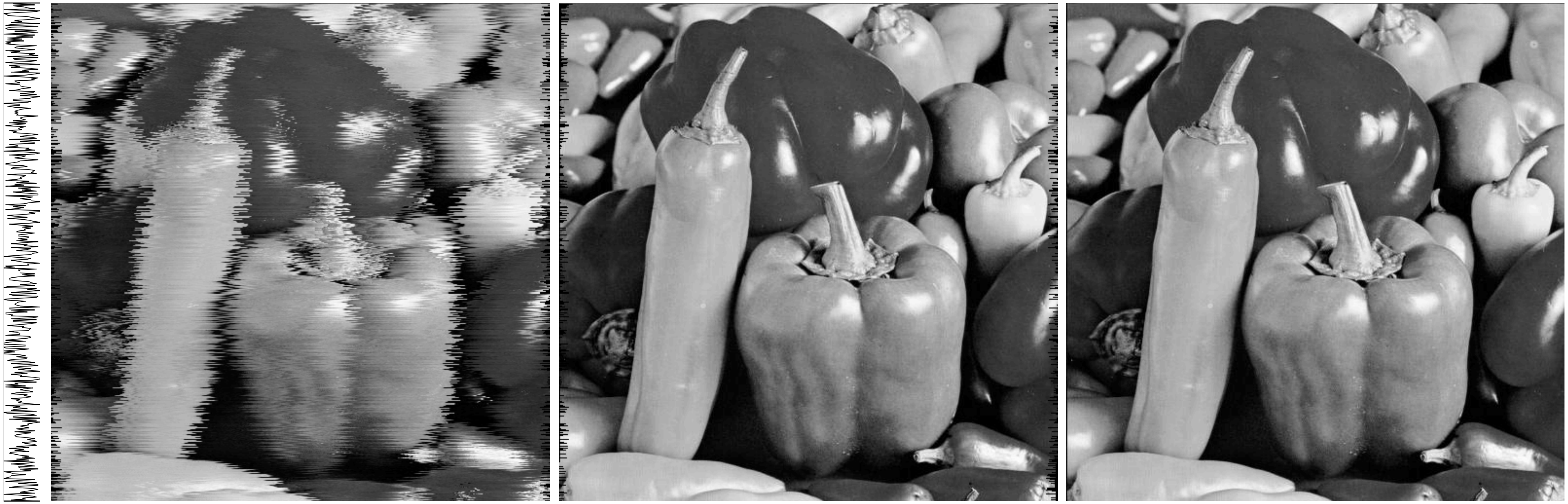
We restore $\hat{\boldsymbol{u}}_i$ using $\hat{\boldsymbol{d}}_i = \arg \min\limits_{|\boldsymbol{d}_i| \leqslant \boldsymbol{N}} \mathcal{F}(\boldsymbol{d}_i)$

$$\mathcal{F}(\boldsymbol{d}_i) = \sum_{j=N+1}^{c-N} \left| \boldsymbol{v}_i(j + \boldsymbol{d}_i) - 2\hat{\boldsymbol{u}}_{i-1}(j) + \hat{\boldsymbol{u}}_{i-2}(j) \right|^{\alpha}, \quad \boldsymbol{\alpha} \in \{0.5, 1\}, \quad \boldsymbol{N} > \boldsymbol{M}$$

piece-wise linear model for the columns of the image

Jittered, $[-20, 20]$ | $\alpha = 1$ | Jitter: $6\sin\left(\frac{n}{4}\right)$ | $\boldsymbol{\alpha = 1} \equiv$ Original

Jitter {-15,..,15} $\alpha = 1,\ \alpha = 0.5$ Original image



Jitter {-10,..,10} $\alpha = 0.5$ Original image

A Monte-Carlo experiment shows that in almost all cases, $\boldsymbol{\alpha = 0.5}$ is the best choice.

The proposed method outperforms by far all other methods *[Kokaram98,Laborelli03,Shen04,Kang06,Scherzer11]*

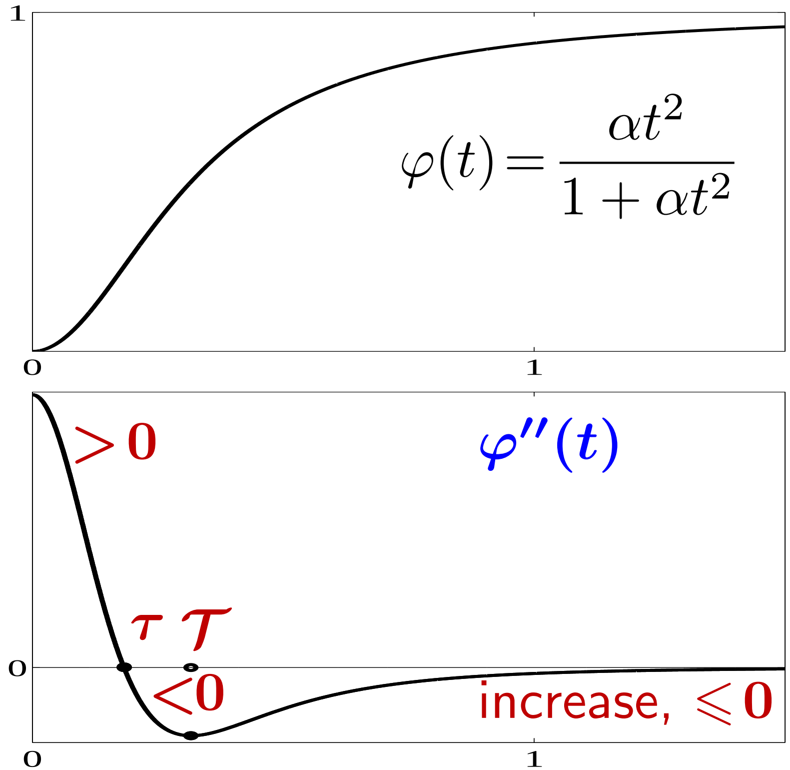
5 Non-convex regularization

[Nikolova 04,10]

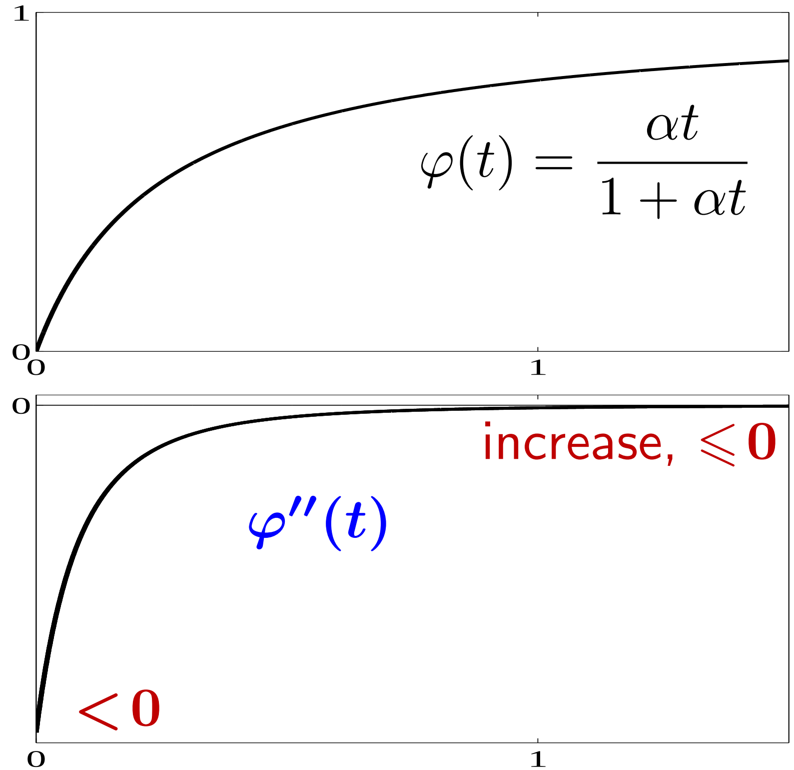
$$\mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|)$$

Standard assumptions on φ: $\mathcal{C}^2$ on $\mathbb{R}_+$ and $\lim_{t\to\infty} \varphi''(t) = 0$, as well as:

$\varphi'(0) = 0$ (Φ is smooth)



$\varphi'(0^+) > 0$ (Φ is nonsmooth)



5.1 Either shrinkage or enhancement of differences

If $\hat{\boldsymbol{u}}$ is a (local) minimizer of $\boldsymbol{\mathcal{F}_v}$ then $\exists\boldsymbol{\theta_0} \geqslant \mathbf{0}, \;\; \exists\boldsymbol{\theta_1} > \boldsymbol{\theta_0}$ so that for $\beta > \mathsf{K}_{(\mathsf{A},\mathsf{G},\varphi)}$

$$\begin{aligned} \widehat{\boldsymbol{h}}_{\mathbf{0}} &= \{\boldsymbol{i} : \|\boldsymbol{G_i}\hat{\mathbf{u}}\| \leqslant \boldsymbol{\theta_0}\} && \textbf{homogeneous regions} \\ \widehat{\boldsymbol{h}}_{\mathbf{1}} &= \{\boldsymbol{i} : \|\boldsymbol{G_i}\hat{\mathbf{u}}\| \geqslant \boldsymbol{\theta_1}\} && \textbf{edges} \\ \widehat{\boldsymbol{h}}_{\mathbf{0}} \cup \widehat{\boldsymbol{h}}_{\mathbf{1}} &= \{\mathbf{1}, \ldots, \boldsymbol{r}\} \end{aligned}$$

Bounds for $(\boldsymbol{\theta_0}, \boldsymbol{\theta_1})$

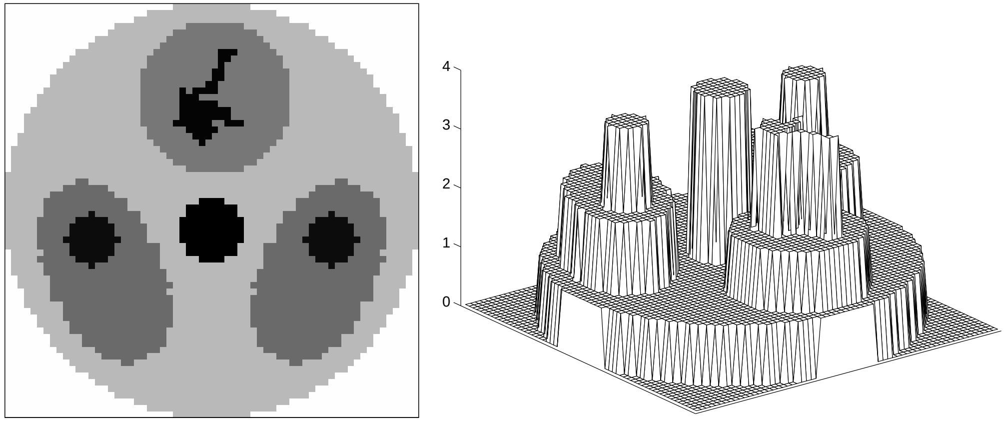
True if $\hat{u}$ is a global minimizer for $\varphi(t) = \min\{\alpha t^2,\ 1\}$ and explicit formula for (θ_0, θ_1)

In particular

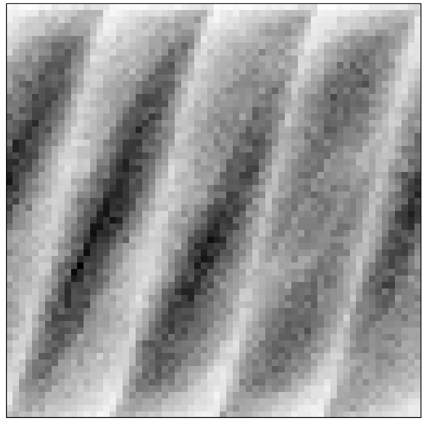
$$\boxed{\boldsymbol{\varphi'(0^+) > 0 \;\;\Rightarrow\;\; \theta_0 = 0} \quad \text{fully segmented image} \quad (G_i\hat{u} = 0,\ \forall i \in \widehat{h}_0)}$$

This holds true if $\hat{u}$ is global minimizer for $\varphi(0) = 0$, $\varphi(t) = 1$ if $t \neq 0$, $\;\theta_1$ explicit

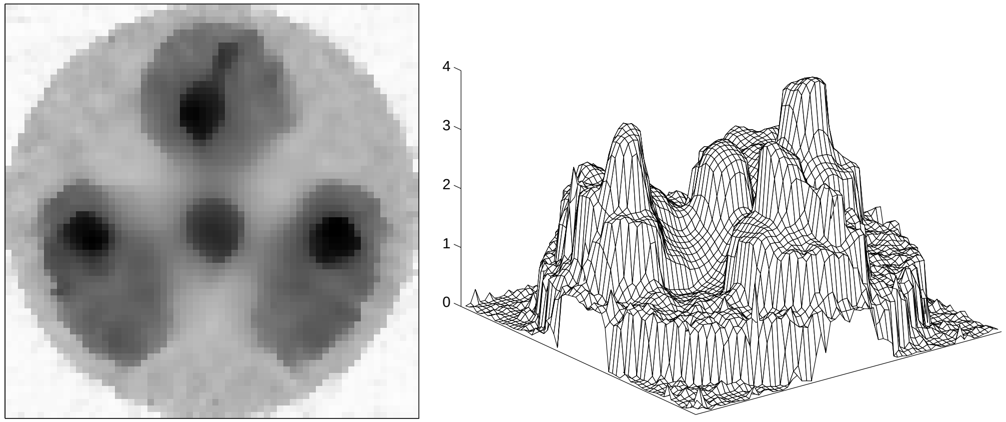
Image Reconstruction in Emission Tomography



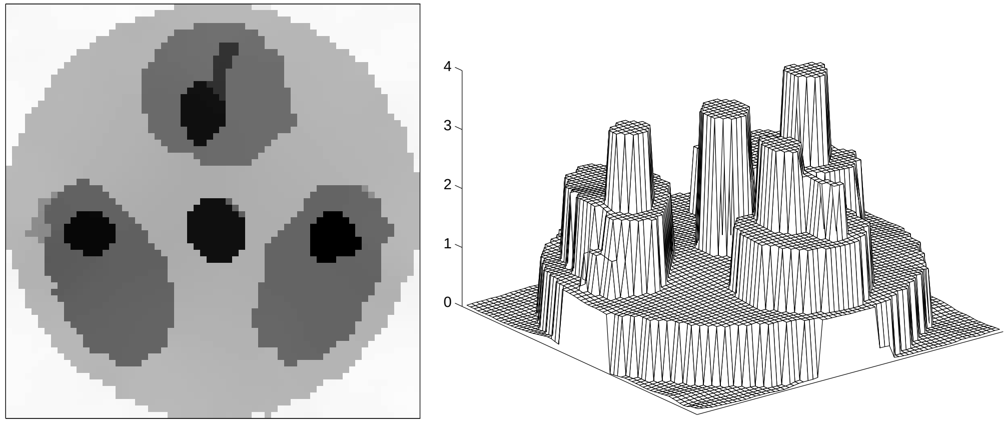
Original phantom



Emission tomography simulated data



φ is smooth (Huber function)



$\varphi(t) = t/(\alpha + t)$ (non-smooth, non-convex)

Reconstructions using $\mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{j \in \mathcal{N}_i} \varphi(|u[i] - u[j]|)$, $\quad \Psi =$ smooth, convex

5.2 Selection for the global minimizer

Additional assumptions: $\|\varphi\|_\infty < \infty$, $\{G_i\}$—1^{st}-order differences, A^*A invertible

$$\mathbb{1}_\Sigma[i] = \begin{cases} 1 & \text{if } i \in \Sigma \subset \{1,..,p\} \\ 0 & \text{else} \end{cases}$$

Original: $u_o = \xi \mathbb{1}_\Sigma, \quad \xi > 0$

Data: $v = \xi \, A \, \mathbb{1}_\Sigma = A u_o$

$$\hat{u} = \textbf{global} \text{ minimizer of } \mathcal{F}_v$$

$\exists \; \xi_1 > 0$ such that $\xi > \xi_1 \quad \Rightarrow \quad \hat{u}$—perfect edges

Moreover:

- $\varphi'(0^+) > 0$, $\varphi'(t) > 0$, $\forall t > 0$, then $\quad \xi > \xi_1 \quad \Rightarrow \quad \hat{u} = c \, u_o, \quad c < 1, \; \lim_{\xi \to \infty} c = 1$

- $\varphi(t) = \eta, \; t \geqslant \eta$, then $\xi > \xi_1 \quad \Rightarrow \quad \hat{u} = u_o$

This holds true also for $\varphi(t) = \min\{\alpha t^2, \; 1\}$ and for $\varphi(0) = 0$, $\varphi(t) = 1$ if $t \neq 0$

6. Nonsmooth data-fidelity and regularization

Consequence of §3 and §4: if Φ and Ψ non-smooth, $\begin{cases} \boldsymbol{G_i \hat{u} = 0} & \text{for} \quad \boldsymbol{i \in \hat{h}_\varphi \neq \varnothing} \\ \boldsymbol{a_i \hat{u} = v[i]} & \text{for} \quad \boldsymbol{i \in \hat{h}_\psi \neq \varnothing} \end{cases}$

6.1 The L_1-TV energy

T. F. Chan and S. Esedoglu, "Aspects of Total Variation Regularized L^1 Function Approximation", SIAM J. on Applied Mathematics, 2005

$$\mathcal{F}_v(u) = \|u - 1\!\!1_\Omega\|_1 + \beta \int_{\mathbb{R}^d} \|\nabla u(x)\|_2 \, dx \quad \text{where} \quad 1\!\!1_\Omega(x) \doteq \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \quad\quad \text{else} \end{cases}$$

- $\exists\, \hat{u} = 1\!\!1_\Sigma$ $\quad$ (Ω convex $\Rightarrow$ $\quad \Sigma \subset \Omega$ and $\hat{u}$ unique for almost every $\beta > 0$)
- contrast invariance: if $\hat{u}$ minimizes for $v = 1\!\!1_\Omega$ then $c\hat{u}$ minimizes $\mathcal{F}_{cv}$

  the contrast of image features is more important than their shapes
- critical values β^* $\begin{cases} \beta < \beta^* & \Rightarrow \quad \text{objects in } \hat{u} \text{ with good contrast} \\ \beta > \beta^* & \Rightarrow \quad \text{they suddenly disappear} \end{cases}$

  $\Rightarrow$ data-driven scale selection

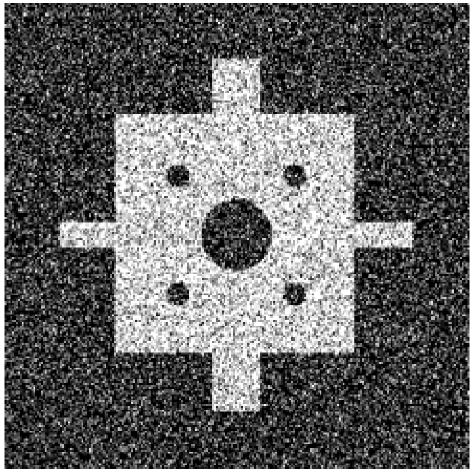
6.2 Binary images by L1 data-fitting and TV *[T. Chan, S. Esedoglu, Nikolova 06]*

Classical approach to find a binary image $\hat{\boldsymbol{u}} = \mathbb{1}_{\hat{\boldsymbol{\Sigma}}}$ from binary data $\mathbb{1}_{\Omega}$, $\Omega \subset \mathbb{R}^2$

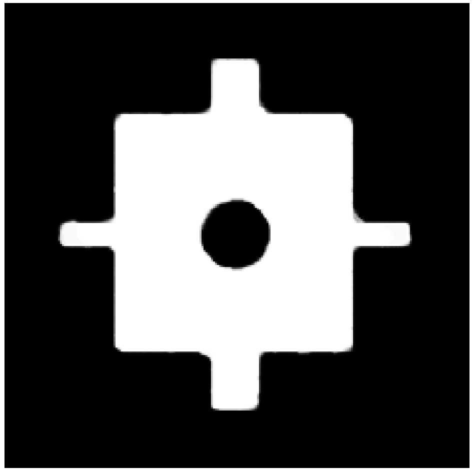
$$\hat{\Sigma} = \arg\min_{\Sigma} \left\{ \left\| \mathbb{1}_{\Sigma} - \mathbb{1}_{\Omega} \right\|_2^2 + \beta \mathrm{TV}(\mathbb{1}_{\Sigma}) \right\} \qquad \text{nonconvex problem} \qquad (\star)$$

usual techniques (curve evolution, level-sets) fail

$\hat{\Sigma}$ solves $(\star)$ $\Leftrightarrow$ $\hat{\boldsymbol{u}} = \mathbb{1}_{\hat{\boldsymbol{\Sigma}}}$ minimizes $\left\| u - \mathbb{1}_{\Omega} \right\|_1 + \beta \, \mathrm{TV}(u)$ (convex)



Data



Restored

6.3 Multiplicative noise removal on Frame coefficients

[Durand, Fadili, Nikolova 09]

Multiplicative noise arises in various active imaging systems e.g. synthetic aperture radar

Original image $S_o \Rightarrow \Sigma_k = S_o \eta_k \Rightarrow$ Data $\Sigma = \dfrac{1}{K}\sum_{k=1}^{K} \Sigma_k = S_o \dfrac{1}{K}\sum_{k=1}^{K} \eta_k = S_o\,\eta$

$\mathrm{pdf}(\eta) =$ Gamma density

We use Log-data $\quad v = \log\Sigma = \log S_o + \log\eta = u_0 + n$

Frame Coefficients: $\quad y = Wv = Wu_0 + Wn$

Hard Thresholding: $\quad y_T[i] = \begin{cases} 0 & \text{if } |y[i]| \leqslant T, \\ y[i] & \text{otherwise} \end{cases} \qquad \forall i \in I, \quad T > 0$ (suboptimal).

$$I_1 = \{i \in I \;:\; |y[i]| > T\} \;\text{ and }\; I_0 = I \setminus I_1$$

A convex $\mathcal{F}_y$ is built using the properties of nonsmooth data-fidelity and regularization.

Restored coefficients $\hat{x} = \arg\min_{x} \mathcal{F}_y(x)$

$$\mathcal{F}_y(x) = \lambda_0 \sum_{i \in I_0} \big|x[i]\big| + \lambda_1 \sum_{i \in I_1} \big|x[i] - y[i]\big| + \|\widetilde{W} x\|_{\mathrm{TV}}$$

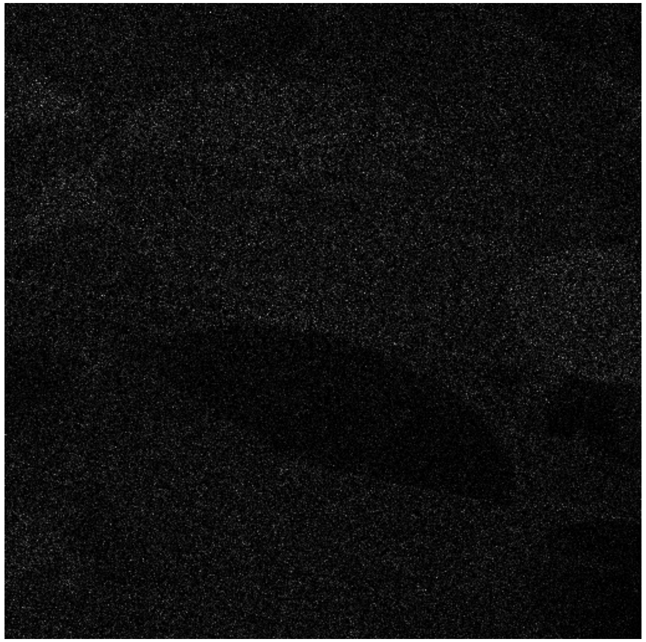
$\hat{S} = B \exp\left(\widetilde{W} \hat{x}\right)$, where $\widetilde{W}$ left inverse, B bias correction

Fast minimization using Douglas-Rachford splitting algorithm

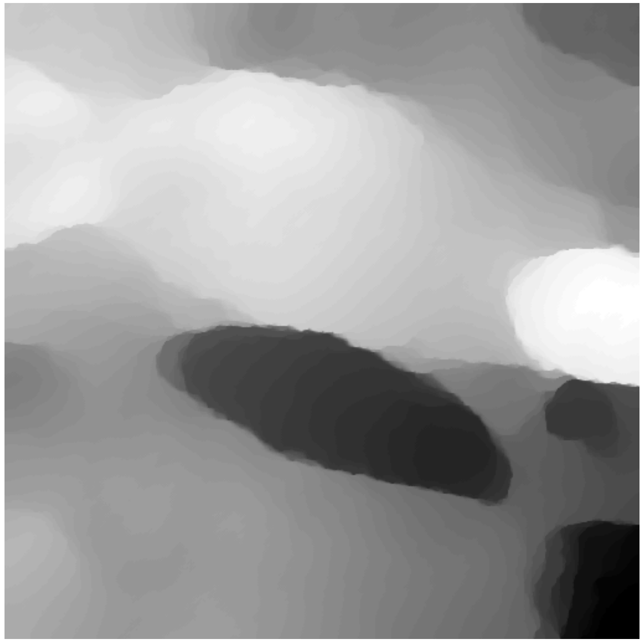
Comparisons

- **BS** *[Chesneau,Fadili,Starck 08]*: Block-Stein thresholds the curvelet coefficients, $\approx$ minimax(large class of images with additive noises), optimal threshold $\mathfrak{T} = 4.50524$
- **AA** *[Aubert,Aujol 08]*: $\Psi = -$ Log-Likelihood, $\Phi =$TV   ($\mathcal{F}_v \equiv$MAP)
- **SO** *[Shi,Osher 08]*: relaxed inverse scale-space for $\mathcal{F}_v(u) = \|v - u\|_2^2 + \beta \mathrm{TV}(u)$ where v is the log-data. Stops when $k^* = \max\{k \in \mathrm{N} : \mathrm{Var}(u^{(k)} - u_o) \geqslant \mathrm{Var}(n)\}$.

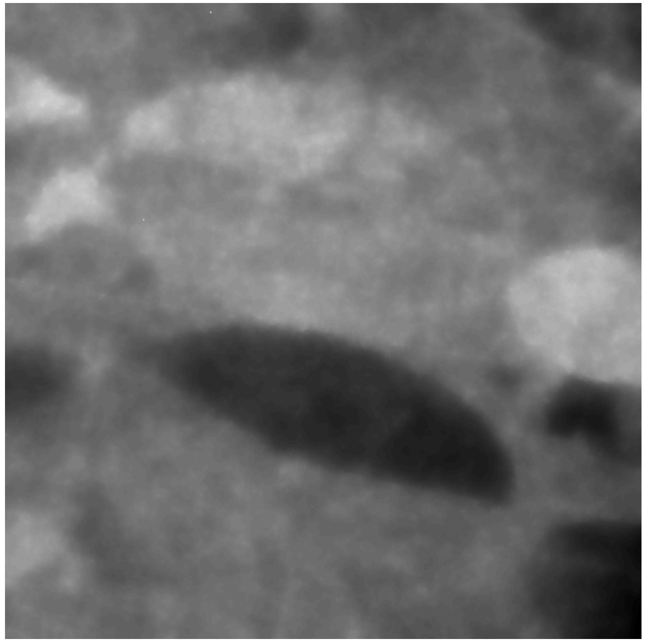
Monte-Carlo comparative experiment confirms the proposed method



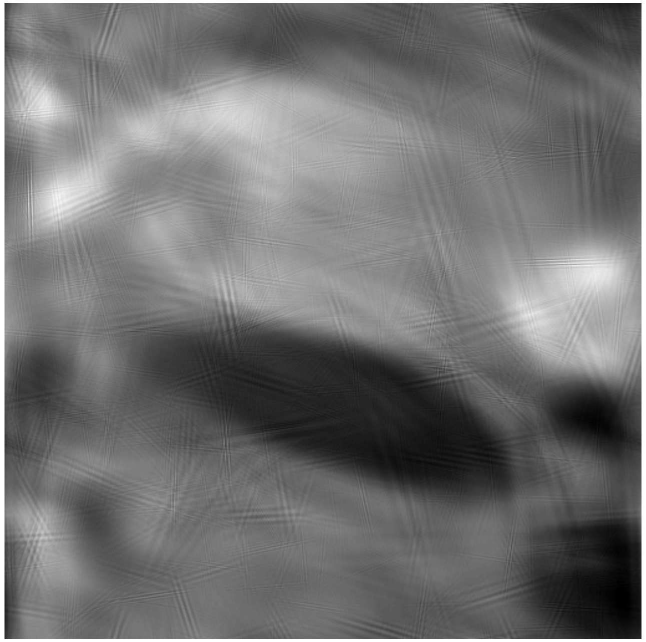
Noisy Fields $K = 1$ (512×512)



SO: PSNR=9.59, MAE=196

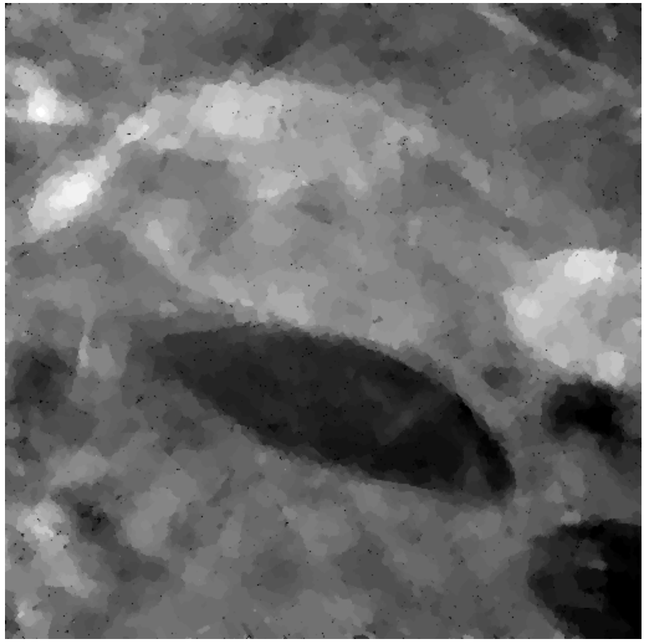


AA: PSNR=15.74, MAE=76.66

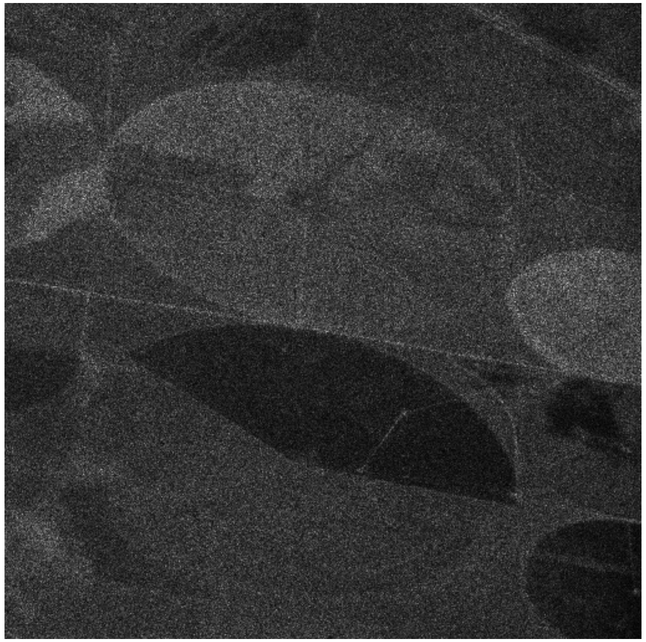


BS: PSNR=22.52, MAE=35.22

Fields (original)

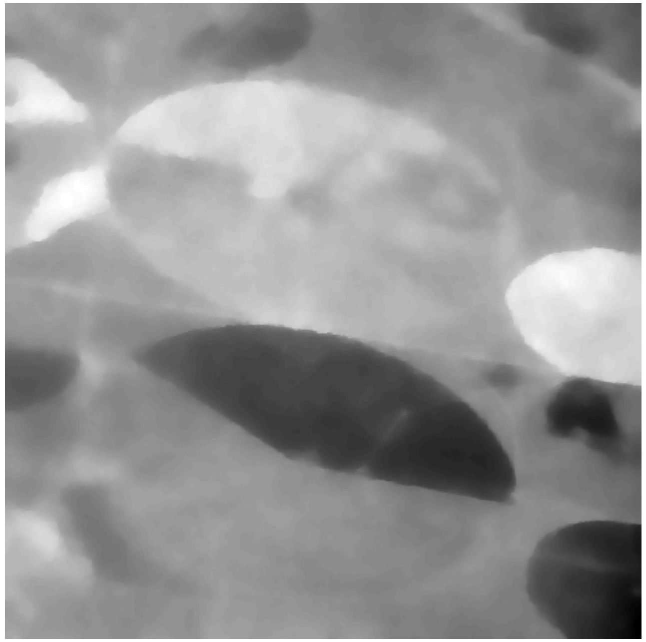


Our: PSNR=22.89, MAE=33.67

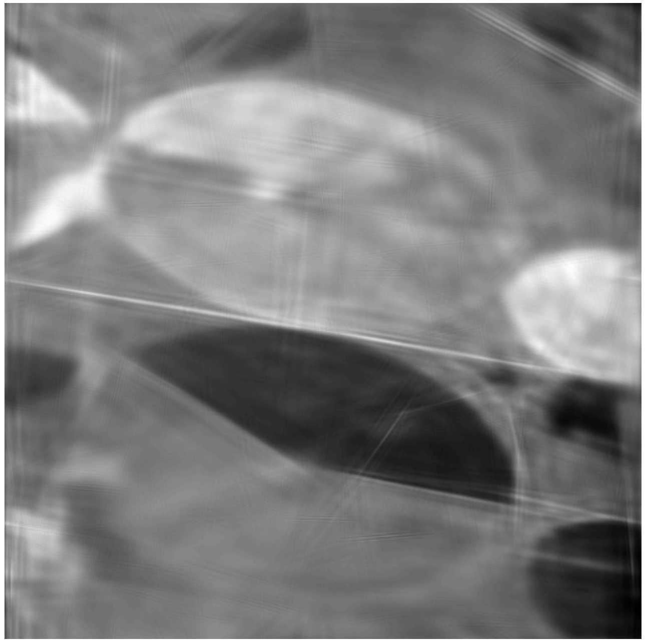


Noisy $K = 10$

SO: PSNR=25.36, MAE=25.14

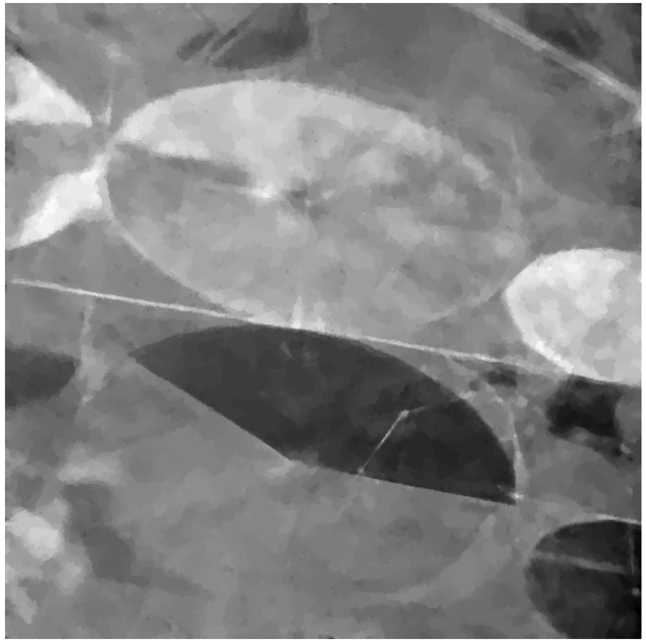


AA: PSNR=17.13, MAE=65.40



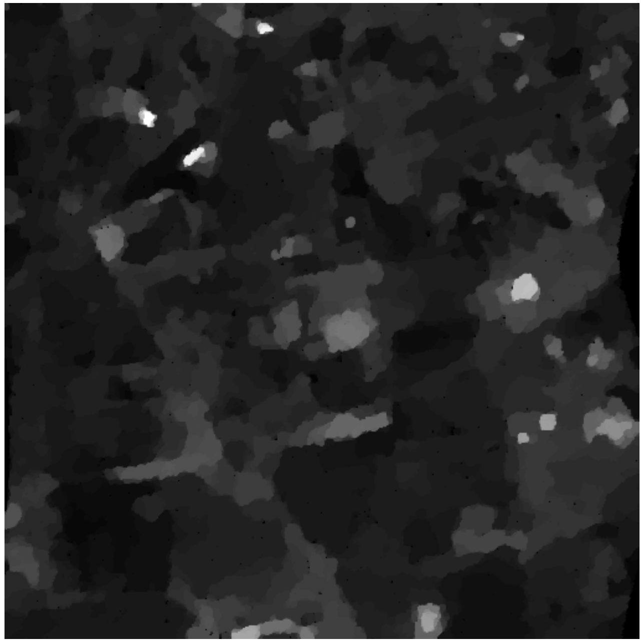
BS: PSNR=27.24, MAE=19.61

Fields (original)



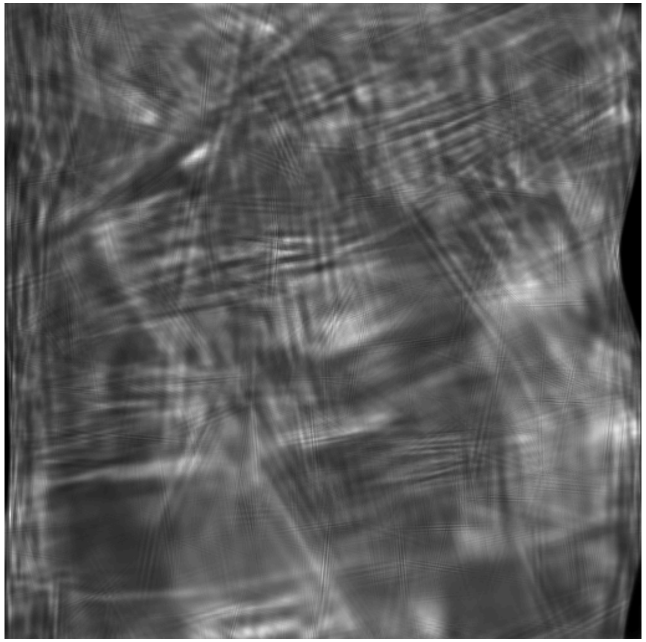
Our: PSNR=28.04, MAE=18.19

Noisy City $K = 1$ (512×512)



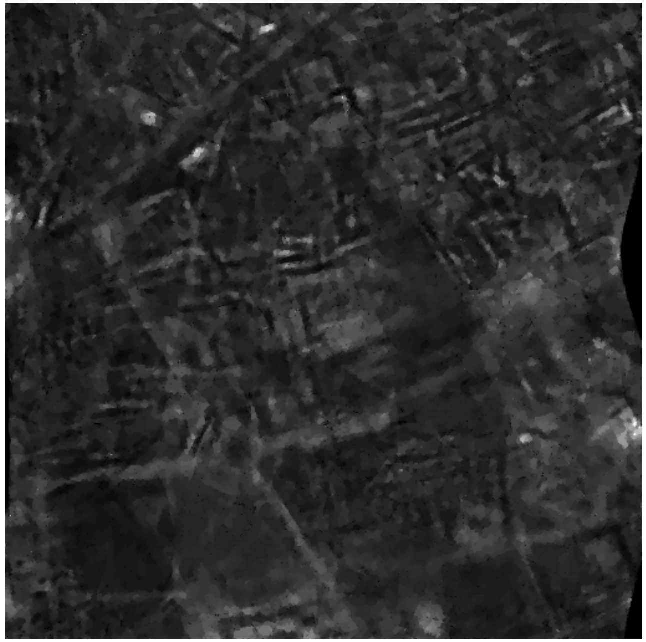
SO: PSNR=18.39, MAE=24.08

AA: PSNR=22.18, MAE=13.71



BS: PSNR=22.25, MAE=13.96

City (original)

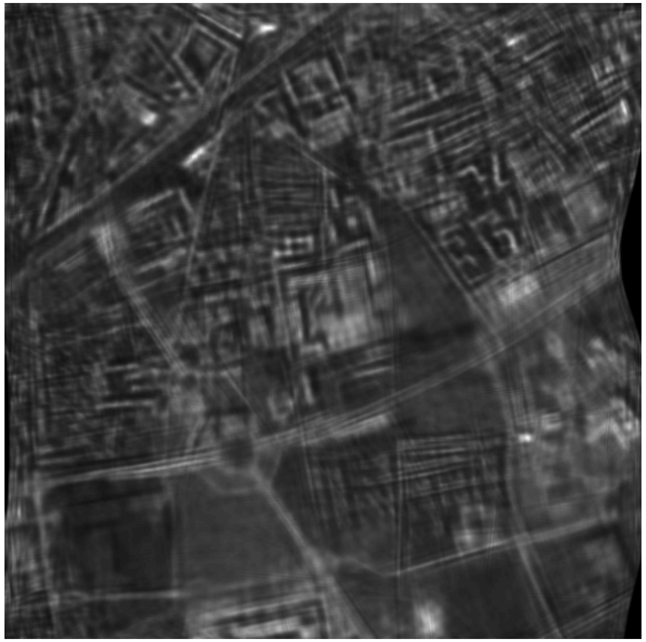


Our: PSNR=22.64, MAE=13.39

Noisy $K = 4$

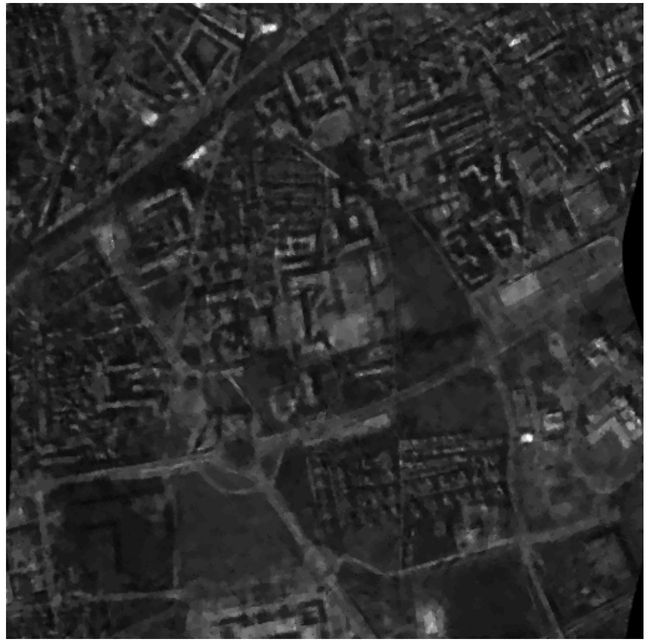
SO: PSNR=24.40, MAE=10.76

AA: PSNR=24.55, MAE=10.06



BS: PSNR=24.92, MAE=9.87

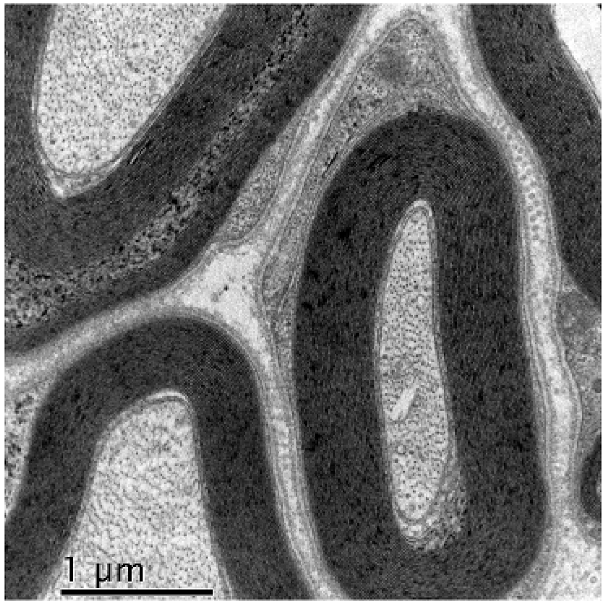
City (original)



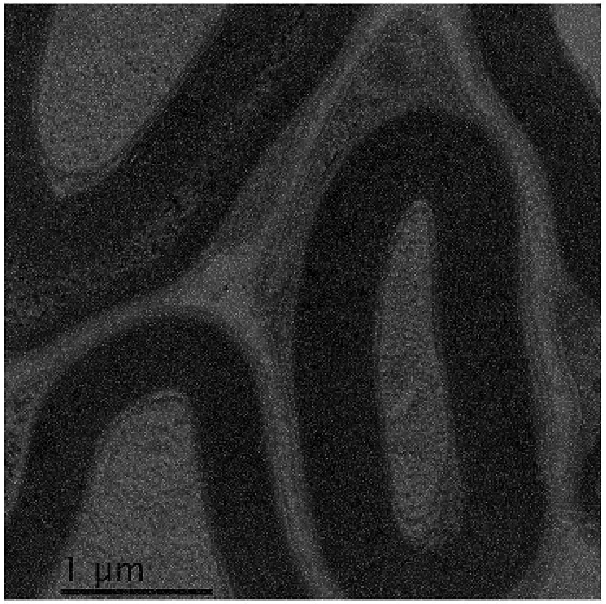
Our: PSNR=25.84, MAE=9.09

C. Clason, B. Jin, K. Kunisch
“Duality-based splitting for fast $\ell_1 - \mathrm{TV}$ image restoration”, 2012,
http://math.uni-graz.at/optcon/projects/clason3/

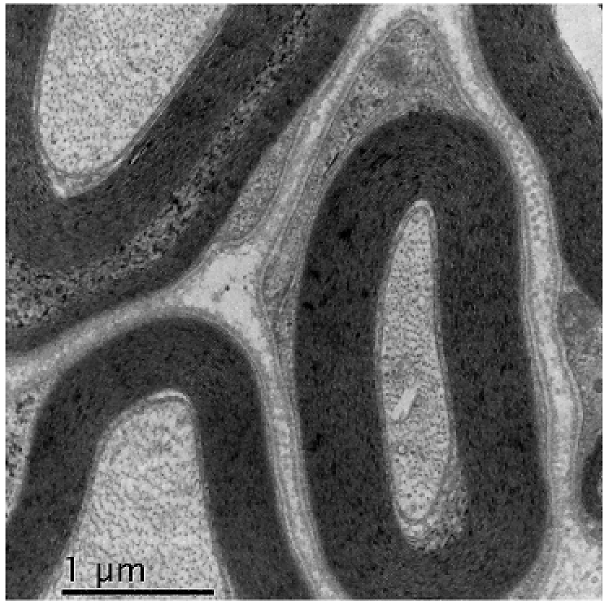
Scanning transmission electron microscopy (2048×2048 image)



true image



noisy image



restoration

6.4 ℓ_1 data-fidelity with concave regularization

[Nikolova, Ng, Tam 12]

$$\mathcal{F}_v(u) = \sum_{i\in I} \big|a_i u - v[i]\big| + \beta \sum_{j \in J} \varphi(\|G_j u\|_2), \quad \varphi'(0^+) > 0, \;\; \varphi''(t) < 0, \; \forall t \geqslant 0$$

$$I = \{1, \cdots, q\}\,, \quad J = \{1, \cdots, r\}$$

No conditions on the rank of the matrix formed by the rows a_i

$\varphi(t)$	$\dfrac{\alpha\, t}{\alpha\, t + 1}$	$1 - \alpha^t, \; \alpha \in (0,1)$	$\ln(\alpha t + 1)$	$(t+\varepsilon)^\alpha, \; \alpha \in (0,1), \; \varepsilon > 0$	$(\cdots)$

This family of objective functions has never been considered before

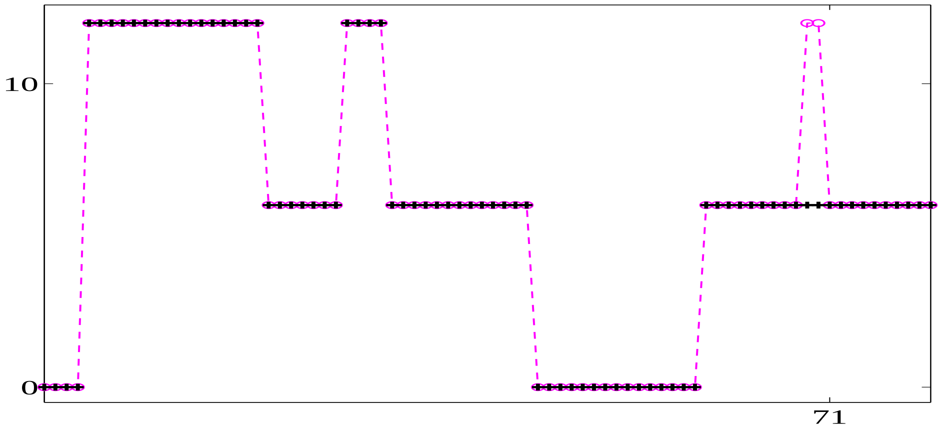
Main feature:

Each pixel of a (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ is involved in (at least) one equation $a_i \hat{u} = v[i]$, or in (at least) one equation $G_j \hat{u} = 0$, or in both types of equations.

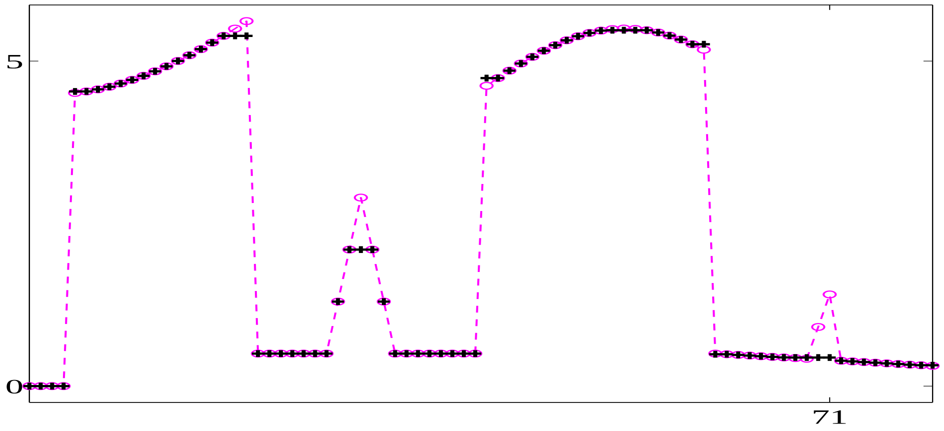
Illustration for: $\mathcal{F}_v(u) = \|u - v\|_1 + \beta \sum_{i=1}^{p-1} \varphi(|u[i+1] - u[i]|)$

$\varphi(t) = \frac{\alpha t}{\alpha t + 1}, \quad \alpha = 4$

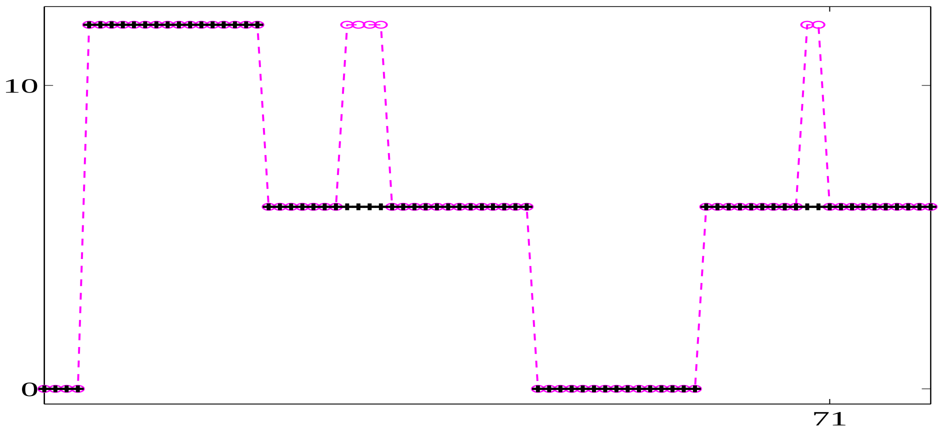
$\varphi(t) = \ln(\alpha t + 1), \quad \alpha = 2$



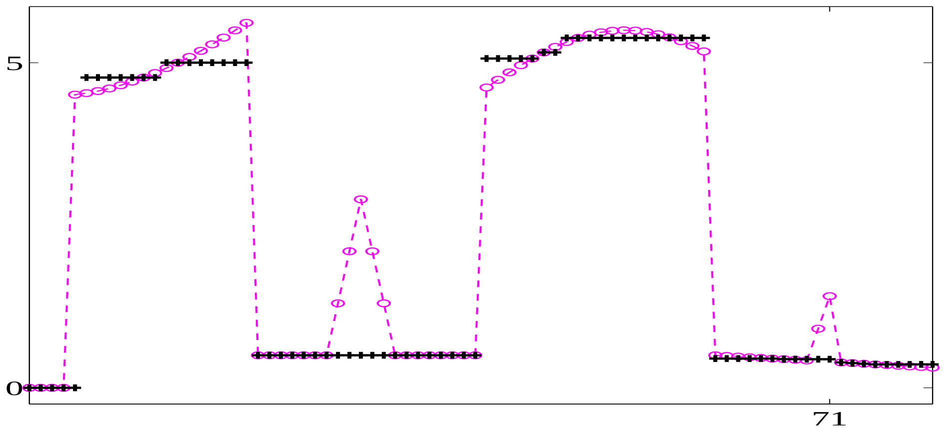
$\beta \in \{78, \cdots, 156\}$



$\beta \in 0.1 \times \{10, \cdots, 14\}$

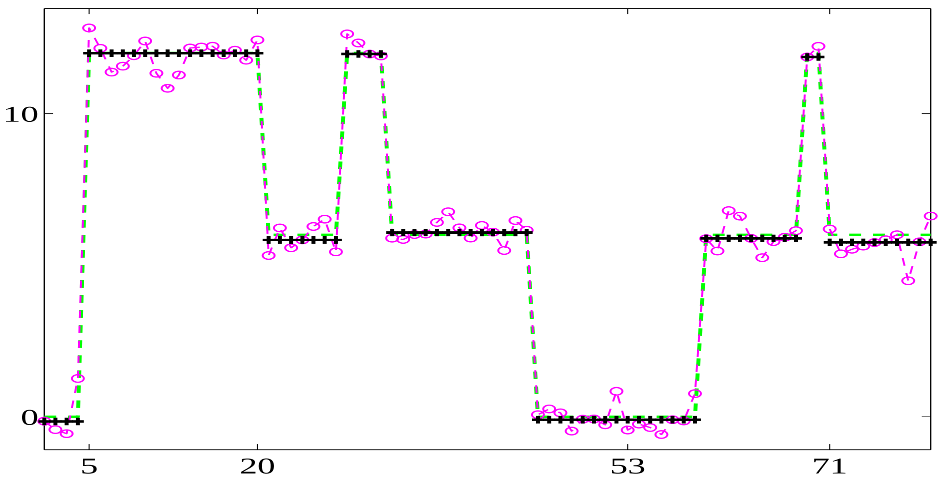


$\beta \in \{157, \cdots, 400\}$

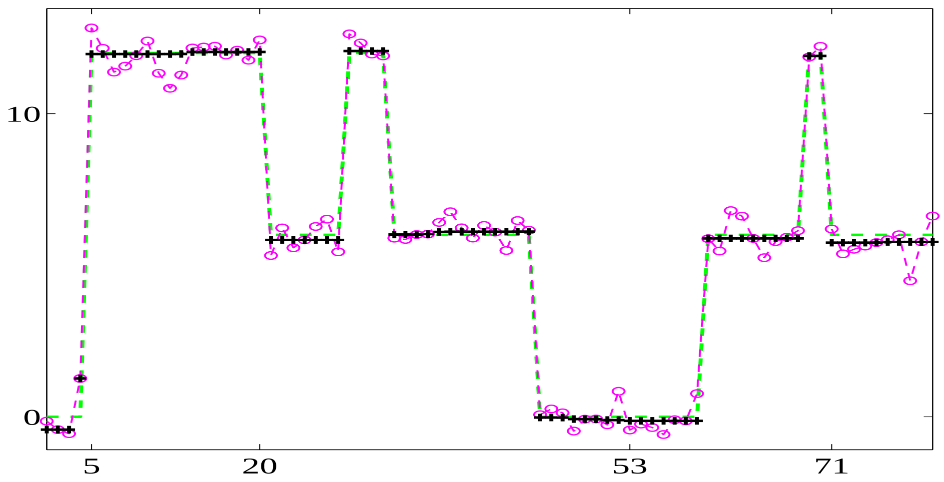


$\beta \in 0.1 \times \{16, \cdots, 30\}$

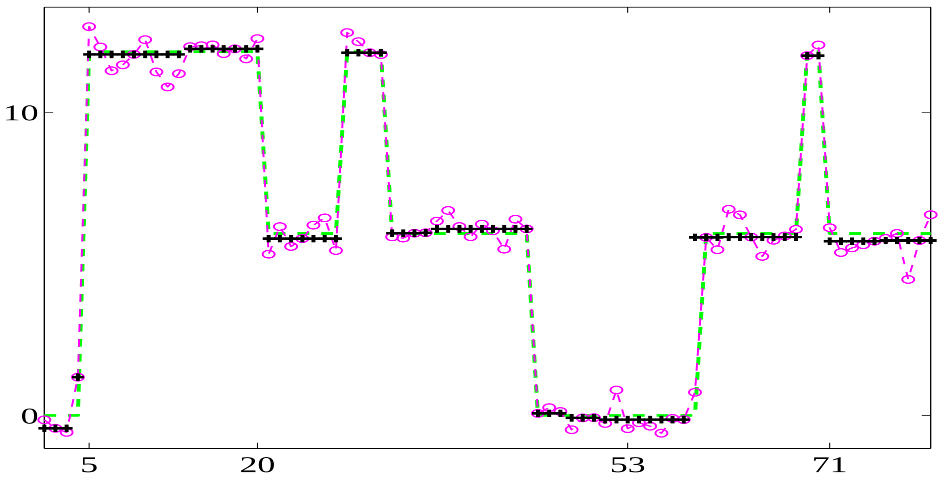
Data samples $v[i]$ (∘∘∘), Minimizer samples $\hat{u}[i]$ (+++).



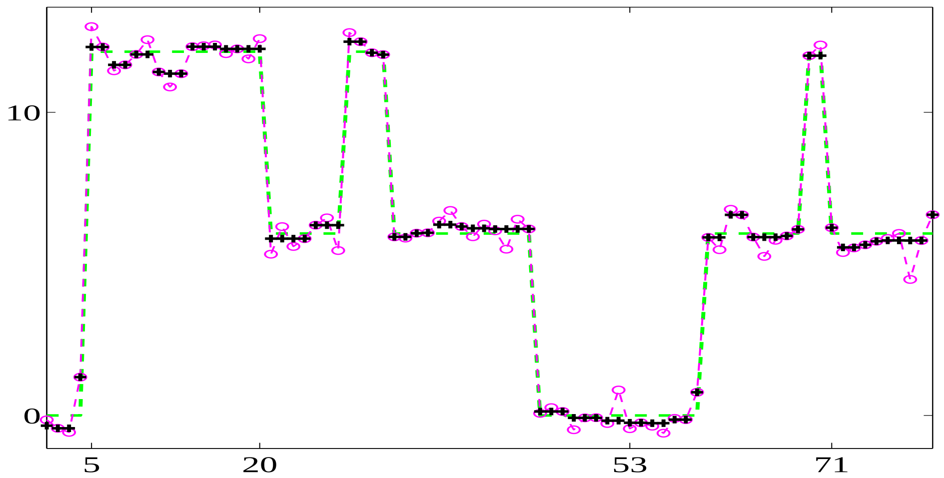
(a) $\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$ $(\alpha = 4)$



(b) $\varphi(t) = 1 - \alpha^t$ $(\alpha = 0.1)$

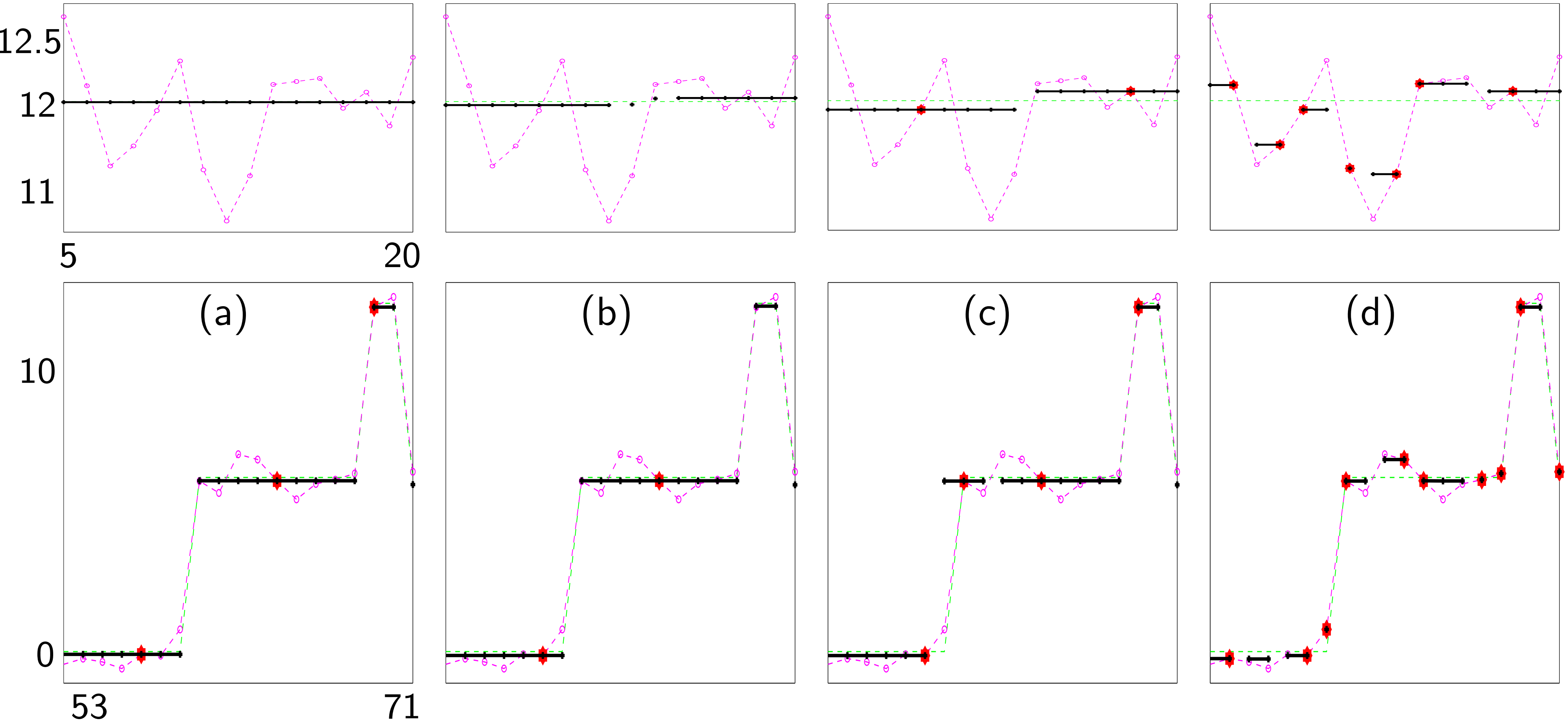


(c) $\varphi(t) = \ln(\alpha t + 1)$ $(\alpha = 2)$



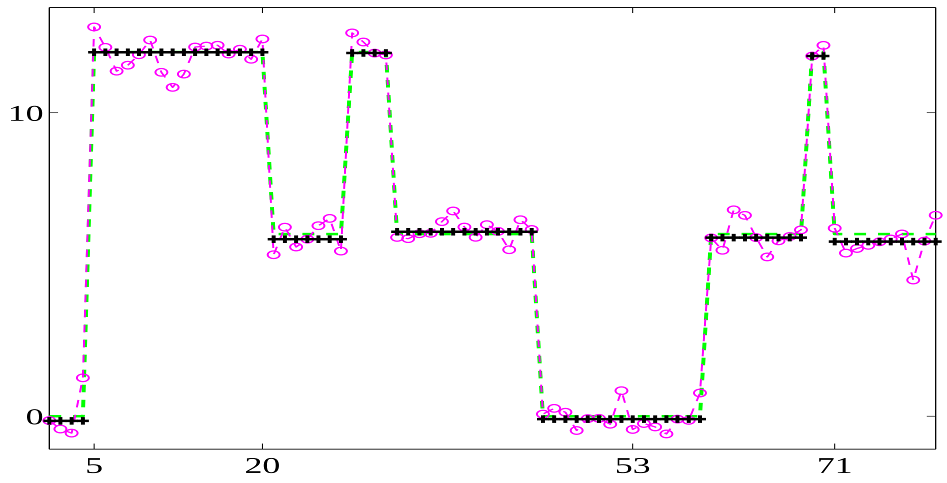
(d) $\varphi(t) = (t + 0.1)^\alpha$ $(\alpha = 0.5)$

Data samples $\boldsymbol{v}[\boldsymbol{i}]$ corrupted with Gaussian noise (ooo). Minimizer samples $\hat{\boldsymbol{u}}[\boldsymbol{i}]$ (+++).
Original $\boldsymbol{u_o}$ (−−−). $\boldsymbol{\beta}$—the largest value so that the gate at 71 survives.

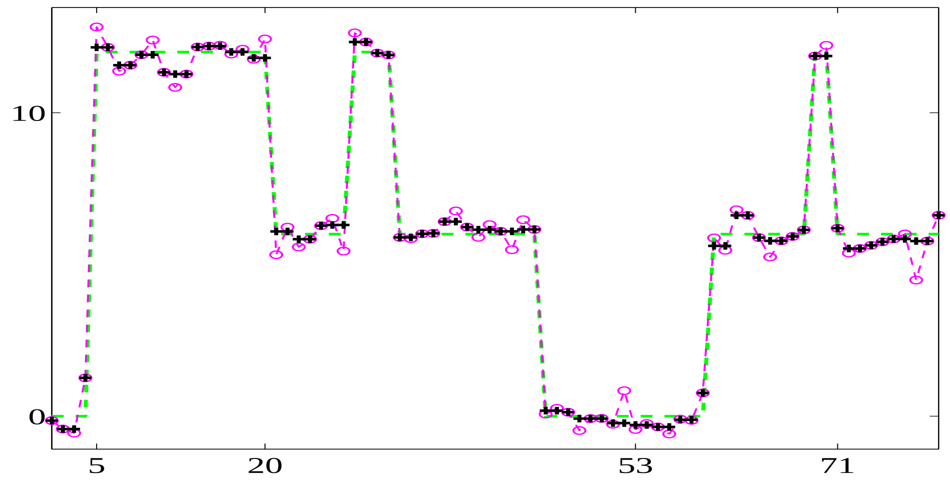


Zooms

Constant pieces—solid black line. Samples meeting $\hat{u}[i] = v[i]$ are flagged with (♦).



$\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$



$\varphi(t) = t \quad (\ell_1 - \mathrm{TV})$

Numerical evidence: critical values $\beta_1, \cdots, \beta_n$ such that

$\beta \in [\beta_i, \beta_{i+1})$ - the minimizer remains unchanged

$\beta \geqslant \beta_{i+1}$ - the minimizer is simplified

- $\mathcal{F}_v$ does have global minimizers

- Let $\hat{u}$ be a (local) minimizer of $\mathcal{F}_v$. Set

$$\begin{aligned} \widehat{I}_0 &= \{i \in I \ : \ a_i \hat{u} = v[i]\} \\ \widehat{J}_0 &= \{j \in J \ : \ G_j \hat{u} = 0\} \end{aligned}$$

$\hat{u}$ is the **unique** point solving the liner system

$$\begin{cases} a_i \hat{u} = v[i] & \forall i \in \widehat{I}_0 \\ G_j \hat{u} = 0 & \forall j \in \widehat{J}_0 \end{cases}$$

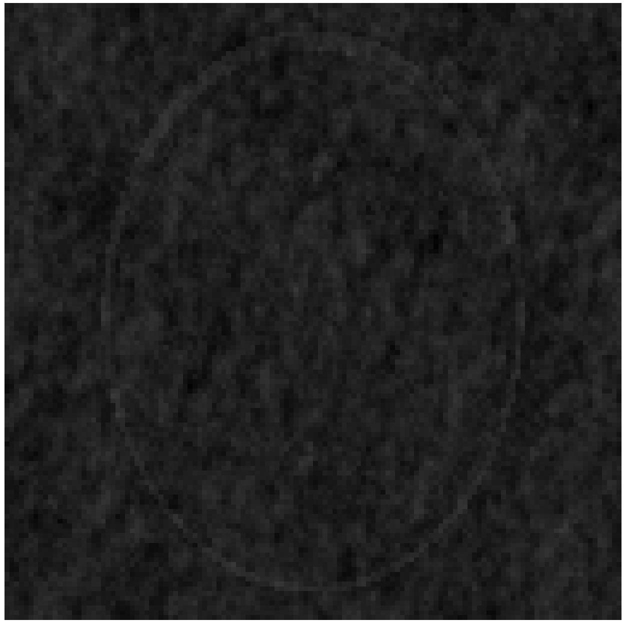
"Contrast invariance" of (local) minimizers

The matrix with rows $\left(a_i, \forall i \in \widehat{I}_0, \ \ G_j, \forall j \in \widehat{J}_0\right)$ has **full column rank**

(a necessary condition for a (local) minimizer)

- All (local) minimizers of $\mathcal{F}_v$ are **strict**

MR Image Reconstruction from Highly Undersampled Data



Zero-filling Fourier recovery

$\|\cdot\|_2^2 + \mathrm{TV}$

$\|\cdot\|_1 + \beta\Phi, \quad \varphi(t) = \frac{\alpha t}{\alpha t+1}$

Reconstructed images from **5%** randomly selected noisy samples in the k-space.

Cartoon

Observed

$\ell_1 - \mathrm{TV}$

$\|\cdot\|_1 + \beta\Phi, \quad \varphi(t) = \frac{\alpha t}{\alpha t+1}$

Knowledge on the features of the minimizers enables
new energies yielding appropriate solutions to be conceived

Imaging science includes research from

- physics
- mathematics
- electrical engineering
- computer vision
- computer science
- perceptual psychology

(WIKIPEDIA)

> “ **We're in Act I of a digital revolution.”**
>
> Jay Cassidy (film editor at `Mathematical Technologies Inc.`)

Thank you!