Average performance of the sparsest approximation in terms of an orthogonal basis

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Our Problem

Data $\boldsymbol{d} \in \mathrm{R}^N$ are uniform on $\boldsymbol{\theta D} = \boldsymbol{B_q^N(\theta)} \stackrel{\text{def}}{=} \{\boldsymbol{v} \in \mathbf{R}^N, \|\boldsymbol{v}\|_q \leq \boldsymbol{\theta}\}$

Orthonormal basis $(\psi_i)_{i \in I}$

The most economical way to represent $\boldsymbol{d} \approx \sum \boldsymbol{\lambda}_i \psi_i$ $=$ solving

$$(\mathcal{P}_d^o) : \quad \begin{cases} \min_{\boldsymbol{\lambda}} \|\boldsymbol{\lambda}\|_0 \\ \text{under the constraint : } \left\| \sum_{i \in I} \boldsymbol{\lambda}_i \psi_i - \boldsymbol{d} \right\| \leq \boldsymbol{\tau} \end{cases}$$

$\|\boldsymbol{\lambda}\|_0 \stackrel{\text{def}}{=} \#\{i \in I : \boldsymbol{\lambda}_i \neq 0\}$ Non-linear approximation

Measure the obtaining of a K-sparse solution

$\equiv$ evaluate $\mathbb{P}\left(\mathrm{val}(\mathcal{P}_d^o) \leq \boldsymbol{K}\right)$

Notation: $\mathrm{val}(\mathcal{P}_d^o) = \|\boldsymbol{\lambda}\|_0$ if $\boldsymbol{\lambda}$ solves $(\mathcal{P}_d^o)$

Related problems

Alternative formulations

$$\min_{\boldsymbol{\lambda}} \|\boldsymbol{\lambda}\|_p \quad \text{subject to} \quad \Big\|\sum_{i\in I}\boldsymbol{\lambda}_i\psi_i - \boldsymbol{d}\Big\| \le \boldsymbol{\tau}, \quad \boldsymbol{p}\in[0,1], \quad \boldsymbol{\tau}\ge 0$$

Minimum description length principle of Rissanen (1967)

$$\|\lambda\|_0 = \sum_{i\in I}\varphi(\lambda_i) \quad \text{where} \quad \varphi(t) = \begin{cases} 0 & \text{if} \quad t=0 \\ 1 & \text{if} \quad t\neq 0 \end{cases}$$

Maximum a posteriori (MAP) energies with Markov-random field prior (signals, images)

$$(\mathcal{E}_d) \qquad \mathcal{E}(\boldsymbol{\lambda}) = \big\|\Psi\boldsymbol{\lambda} - \boldsymbol{d}\big\|_2^2 + \beta\sum_i \varphi(\|D_i\boldsymbol{\lambda}\|), \quad \text{where } D_i \text{ linear operator}$$

(e.g. differences or discrete gradient)

- $\lambda =$ image composed of labels $\Rightarrow$ Potts model *[Geman[2] 1984, Besag 1986,...]*
- $\lambda \in \mathrm{R}^N$—since *[Leclerc 1989]*
- Hard thresholding to denoise wavelet coefficients *[Donoho & Johnstone 1992]*

  $$\text{minimize } \|\boldsymbol{\lambda}_i - \boldsymbol{g}_i\|_2^2 + \beta\varphi(\boldsymbol{\lambda}_i), \ \forall i \in I$$

  where noisy coefficients $\boldsymbol{g}_i = \langle \psi_i, \boldsymbol{d}\rangle, \forall i \in I$
- Theoretical results on $\arg\min \mathcal{E}(\boldsymbol{\lambda})$ in *[Nikolova 2005]*

Nonlinear approximation theory (*K best term approximation.*) *(Review—see [DeVore 1998])*

For $D \subset \mathrm{R}^N$ *and* $d \in D$, *performance is measured by the decay of*

$$K \to \sup_{d \in D} \inf_{J \subset I, \lambda \in \mathrm{R}^J} \left\{ \|d - \sum_{i \in J} \lambda_i \psi_i\| \ : \ \dim(\mathrm{span}(\psi_i)_{i \in J} = K \right\}$$

Performance evaluation only depends on ***D*** *and it is* pessimistic*: it exhibits the* worst case.

Our approach: **Average Performance in Approximation (APA) methodology**

Our focus: **Study the behavior** $\mathbb{P}\left(\mathbf{val}(\boldsymbol{\mathcal{P}^o_d}\ )\right)$
as a function of the Model in a simple context (Basis)

$$\textit{Desideratum: } \mathbb{P}\left(\mathbf{val}(\boldsymbol{\mathcal{P}_d}\ )\right) \begin{cases} \textit{large} \ \ \textit{if } \boldsymbol{K} \textit{ is small} \\ \textit{small} \ \ \textit{if } \boldsymbol{K} \textit{ is large} \end{cases}$$

Context of this work

Tolerance constraint: for $w = \sum_i \lambda_i \psi_i$

$$\|w\| \stackrel{\text{def}}{=} \left(\sum_i |\lambda_i|^p\right)^{\frac{1}{p}}, \; 1 \le p < \infty \quad \text{and} \quad \|w\|_\infty \stackrel{\text{def}}{=} \sup_{1 \le i \le n} |\lambda_i|$$

Note that the numerical problem is easy to solve.

Substitute: $d[i] \leftarrow \langle d, \psi_i \rangle, \; \forall i$. Then

$$(\mathcal{P}_d^o) \iff (\mathcal{P}_d)$$

$$(\mathcal{P}_d): \quad \min_{u \in \mathbf{R}^N} \|u\|_0 \quad \text{subject to}: \quad \|u - d\|_p \le \tau$$

d is uniform on $\theta D = B_q^N(\theta)$ (the θ-radius ℓ_q ball)

1. The Basic Idea

For any $J \subset I$ define $T_J = \mathrm{span}\left\{(e_j)_{j\in J}\right\}$ $(e_j)_{j=1}^N$ the canonical basis of R^N

$$\mathcal{T}_J^\tau = T_J + \left(T_{J^c} \cap B_p^{N-K}(\tau)\right) \qquad K \stackrel{\mathrm{def}}{=} \#J, \quad \#J^c = N-K$$

$\forall\ K \leq N$ we set

$\mathcal{J}(K)$ is a maximal non-redundant listing of all subspaces T_J of dimension K

Theorem

$$\Sigma_K^\tau \stackrel{\mathrm{def}}{=} \left\{d \in \mathbf{R}^N, \mathrm{val}(\mathcal{P}_d) \leq K\right\} = \bigcup_{J\in\mathcal{J}(K)} \mathcal{T}_J^\tau = \bigcup_{J\in\mathcal{J}(K)} T_J + \left(T_{J^c} \cap B_p^{N-K}(\tau)\right)$$

all data $\Rightarrow$ solutions with $\leq K$ coefficients $\neq \mathbf{0}$ (i.e. K-sparse)

Derive tight Upper/Lower bounds for $\Sigma_K^\tau \cap B_q^N(\theta)$

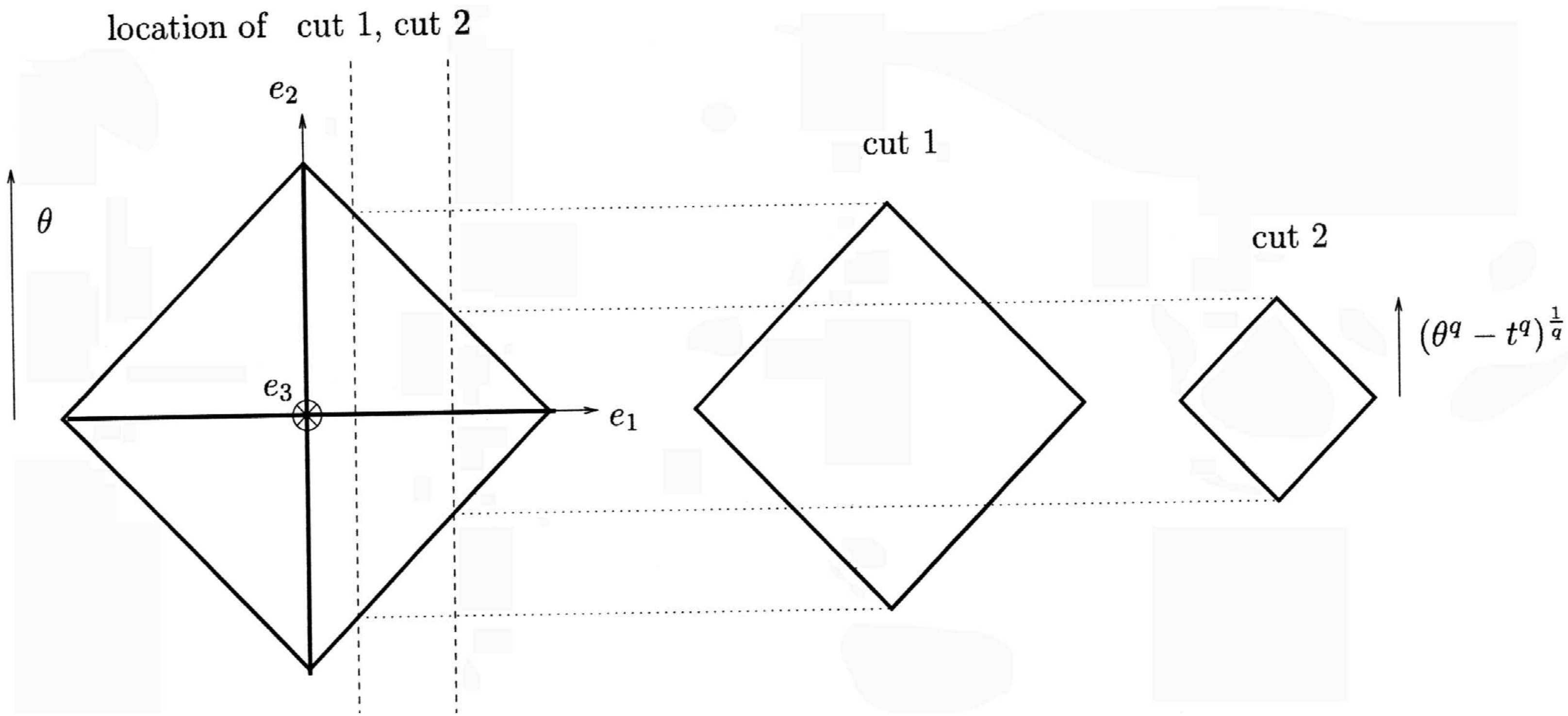
Simulations in typical cases

Deduce bounds on $\mathbb{P}\left(\mathrm{val}(\mathcal{P}_d) \leq K\right)$

2. Preliminary results

The Lebesgue measure of the ℓ_p unit ball on R^n, $1 \le p \le \infty$ $\quad (\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy)$:

$$\alpha_p(n) \stackrel{\text{def}}{=} \mathbb{L}^n(B_p^n(1)) = \frac{\left(2\Gamma\left(1+1/p\right)\right)^n}{\Gamma\left(1+n/p\right)}, \quad \alpha_p(n) \overset{n\to\infty}{\Downarrow} 0 \text{ if } p < \infty$$



The ℓ_1 ball

(industrial drawing)

Cut 2: $u_1 = t > 0$

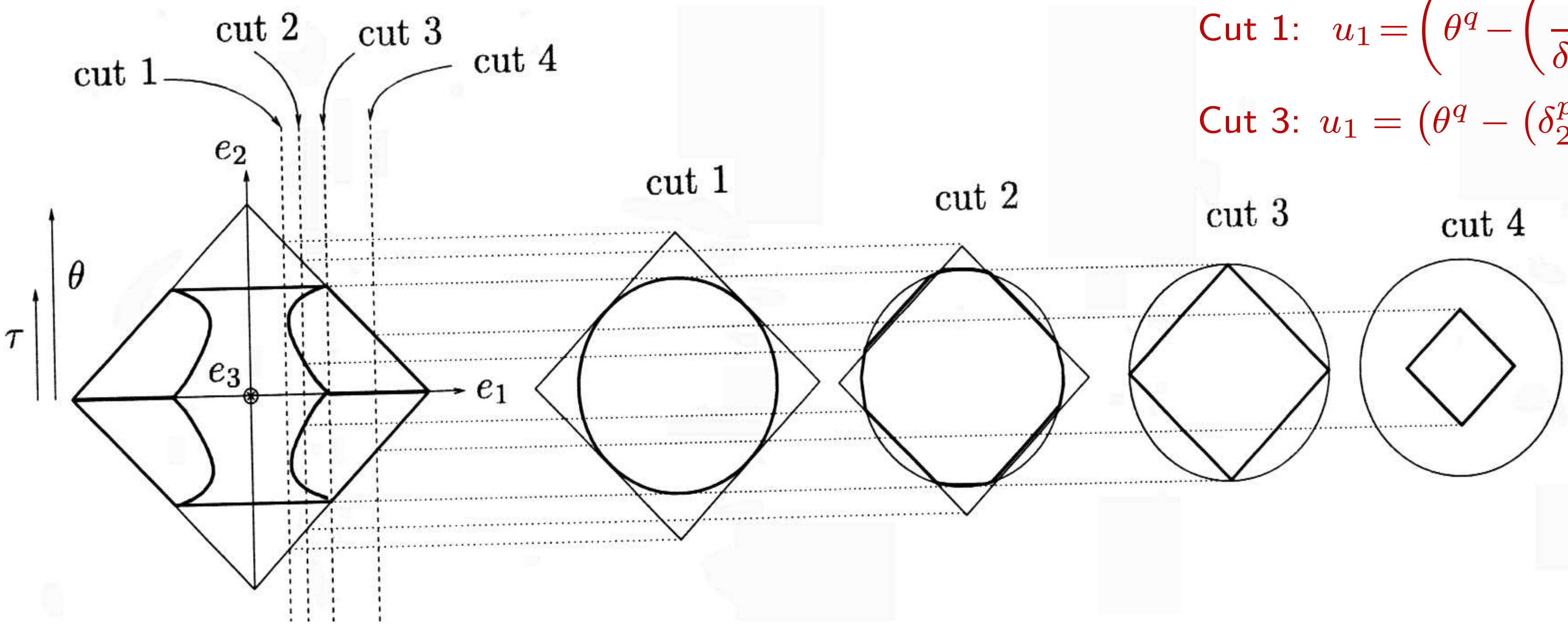
$$\delta_n^{p,q} \stackrel{\text{def}}{=} \begin{cases} n^{\frac{1}{q}-\frac{1}{p}} & \text{if} \quad 1 \le q < p \le \infty \\ 1 & \text{if} \quad 1 \le p \le q \le \infty \end{cases} \Rightarrow \quad \delta_n^{p,q} \ge 1 \text{ and } \begin{cases} \|u\|_q \le \delta_n^{p,q} \|u\|_p \\ \|u\|_p \le \delta_n^{q,p} \|u\|_q \end{cases}$$

We wish to measure sets of the form $\mathcal{T}_J^\tau \cap B_q^N(\theta)$

Industrial drawing: $\mathcal{T}_{\{1\}}^\tau \cap B_1^3(\theta)$ for $p=2$ $\left(\mathcal{T}_{\{1\}}^\tau = \mathrm{span}\{e_1\} + B_2^2(\tau) \right)$

Cut 1: $u_1 = \left(\theta^q - \left(\frac{\tau}{\delta_2^{q,p}}\right)^q\right)^{\frac{1}{q}}$

Cut 3: $u_1 = \left(\theta^q - \left(\delta_2^{p,q}\tau\right)^q\right)^{\frac{1}{q}}$



$\forall J \in \mathcal{J}(K), \quad \forall 1 \le p, q \le \infty \quad \forall \theta > \tau > 0$

$$\mathcal{V}_{N,K}^{p,q}(\tau, \theta) \stackrel{\mathrm{def}}{=} \mathbb{L}^N\left(\mathcal{T}_J^\tau \cap B_q^N(\theta)\right)$$

Below we provide bounds for $\mathcal{V}_{N,K}^{p,q}(\tau, \theta)$

$$\forall t \in \mathbf{R} \quad \mu_q(t) \stackrel{\text{def}}{=} \max\left\{(1-|t|^q)^{\frac{1}{q}},\ 0\right\},\ 1 \le q < \infty \quad \text{and} \quad \mu_\infty(t) \stackrel{\text{def}}{=} 0$$

$$1 \le K \le N-1 \quad \begin{cases} \mathcal{L}_K^{p,q}(\tau,\theta) \stackrel{\text{def}}{=} \displaystyle\int_0^1 {\mu_q}^K \left(\delta_{N-K}^{p,q} \frac{\tau}{\theta} s^{\frac{1}{N-K}}\right) ds \\ \mathcal{U}_K^{p,q}(\tau,\theta) \stackrel{\text{def}}{=} \displaystyle\int_0^1 {\mu_q}^K \left(\frac{1}{\delta_{N-K}^{q,p}} \frac{\tau}{\theta} s^{\frac{1}{N-K}}\right) ds \end{cases} \qquad \mathcal{L}_N^{p,q}(\tau,\theta) \stackrel{\text{def}}{=} \mathcal{U}_N^{p,q}(\tau,\theta) \stackrel{\text{def}}{=} 1$$

$\mathcal{L}_K^{p,q}(\tau,\theta) \in (0,1]$ (lower *correction* factor) $\le \mathcal{U}_K^{p,q}(\tau,\theta) \in (0,1]$ (upper *correction* factor)

Both are the limits of Riemann sums (used in the simulations) and $\longrightarrow 1$ *as* $\frac{\tau}{\theta} \longrightarrow 0$

Theorem

$$\begin{array}{l} J \subset I,\ \theta > \tau > 0 \\ 1 \le p, q \le \infty \end{array} \Rightarrow \begin{cases} \mathcal{V}_{N,K}^{p,q}(\tau,\theta) \ge \alpha_p(N-K)\alpha_q(K) \left(\frac{\tau}{\theta}\right)^{N-K} \theta^N\ \mathcal{L}_K^{p,q}(\tau,\theta) \\ \mathcal{V}_{N,K}^{p,q}(\tau,\theta) \le \alpha_p(N-K)\alpha_q(K) \left(\frac{\tau}{\theta}\right)^{N-K} \theta^N\ \mathcal{U}_K^{p,q}(\tau,\theta) \end{cases}$$

In particular:

$$\begin{cases} 1 \le p = q < \infty & \Rightarrow \quad \mathcal{U}_K^{p,q}(\tau,\theta) = \mathcal{L}_K^{p,q}(\tau,\theta) \\ q = \infty & \Rightarrow \quad 1 = \mathcal{U}_K^{p,q}(\tau,\theta) = \mathcal{L}_K^{p,q}(\tau,\theta) \end{cases}$$

3. How to measure $\boldsymbol{\Sigma_K^\tau \cap B_q^N(\theta) = \bigcup_{J \in \mathcal{J}(K)} \left(\mathcal{T}_J^\tau \cap B_q^N(\theta)\right)}$

Exact measure of $\Sigma_K^\tau \cap B_q^N(\theta)$ needs we decompose it into disjoint sets and then measure the sum of the latter. It means we subtract from $\Sigma_K^\tau \cap B_q^N(\theta)$ multiple intersections like $\mathcal{T}_{J_1}^\tau \cap \mathcal{T}_{J_2}^\tau \cap B_q^N(\theta)$, $\mathcal{T}_{J_1}^\tau \cap \mathcal{T}_{J_2}^\tau \cap \mathcal{T}_{J_3}^\tau \cap B_q^N(\theta)$, and so on, for $J_1, J_2, J_3 ... \in \mathcal{J}(K)$, and measure the result:

this is an open question

Instead we provide computable lower and upper bounds for $\mathbb{L}^N\left(\boldsymbol{\Sigma_K^\tau \cap B_q^N(\theta)}\right)$

Hopefully: multiple intersections of increasing order become smaller and smaller

4. A straightforward upper bound

Theorem

$$\mathbb{L}^N\big(\Sigma_K^\tau \cap B_q^N(\theta)\big) \leq \mathcal{C}_N^K \mathcal{V}_{N,K}^{p,q}(\tau,\theta)$$

Arguments:

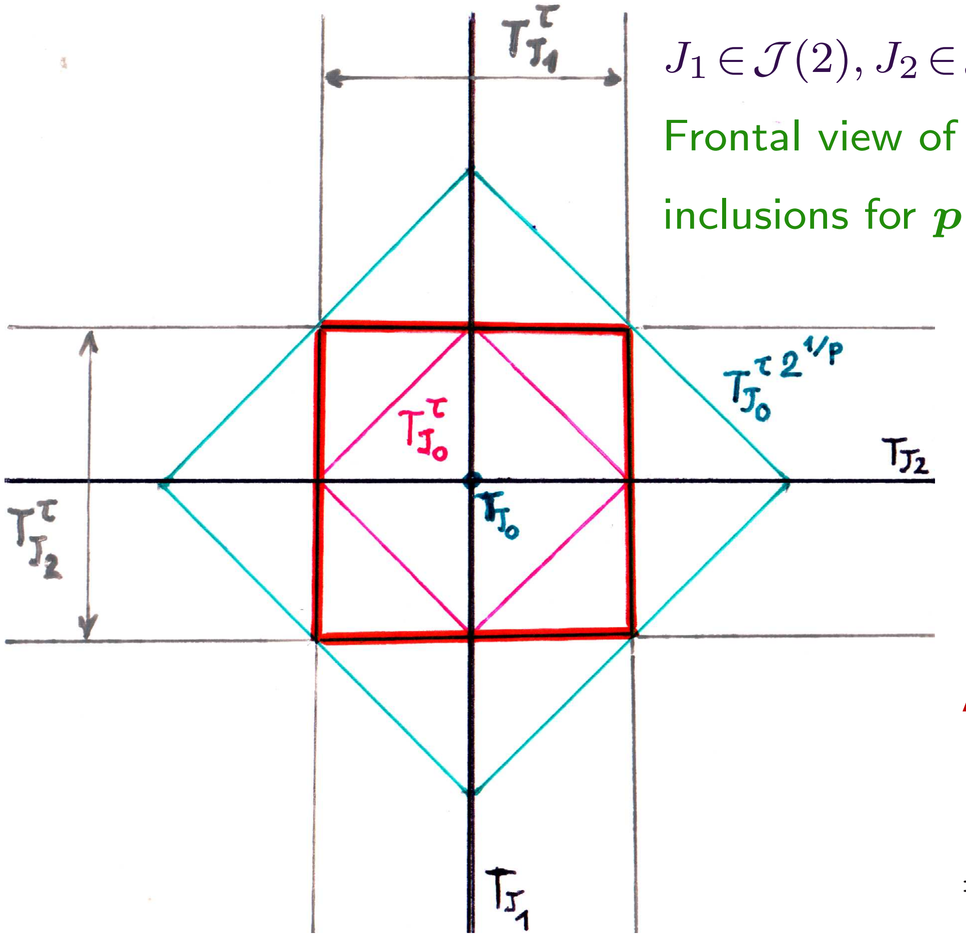
- $\mathbb{L}^N\big(\Sigma_K^\tau \cap B_q^N(\theta)\big) \leq \sum_{J\in\mathcal{J}(K)} \mathbb{L}^N\big(\mathcal{T}_J^\tau \cap B_q^N(\theta)\big)$
- $\mathbb{L}^N\big(\mathcal{T}_J^\tau \cap B_q^N(\theta)\big) = \mathcal{V}_{N,K}^{p,q}(\tau,\theta)$ whenever $\#J = K$
- $\mathcal{J}(K)$ contains $\mathcal{C}_N^K = \frac{N!}{K!(N-K)!}$ elements

5. Derivation of a lower bound

Second-order intersections

Lemma

$$(\boldsymbol{J_1}, \boldsymbol{J_2}) \subset \boldsymbol{\mathcal{J}(K)^2} \quad \Rightarrow \quad \boldsymbol{\mathcal{T}^{\tau}_{J_0}} \subset \left(\boldsymbol{\mathcal{T}^{\tau}_{J_1}} \cap \boldsymbol{\mathcal{T}^{\tau}_{J_2}} \right) \subset \boldsymbol{\mathcal{T}^{\tau 2^{\frac{1}{p}}}_{J_0}} \quad \text{where} \quad \boldsymbol{J_0} \overset{\textbf{def}}{=} \boldsymbol{J_1 \cap J_2}$$



$J_1 \in \mathcal{J}(2), J_2 \in \mathcal{J}(2)$

Frontal view of the inclusions for $\boldsymbol{p = 1}$

These inclusions are sharp but yield loose measure bounds (especially if $p = 1$)

The precise evaluation of
$$\boldsymbol{\mathbb{L}^N(\mathcal{T}^{\tau}_{J_1} \cap \mathcal{T}^{\tau}_{J_2} \cap B^N_q(\theta))}$$
remains an open problem

Note that $\mathcal{T}^{\tau}_{J_0} = \left(\mathcal{T}^{\tau}_{J_1} \cap \mathcal{T}^{\tau}_{J_2} \right)$ if $p = \infty$

A key consequence: for $J_1, J_2 \in \mathcal{J}(K)$ we have

$$\boldsymbol{(\mathcal{T}^{\tau}_{J_1} \setminus \Sigma^{\tau 2^{\frac{1}{p}}}_{K-1}) \cap (\mathcal{T}^{\tau}_{J_2} \setminus \Sigma^{\tau 2^{\frac{1}{p}}}_{K-1})} =$$
$$= \left(\mathcal{T}^{\tau}_{J_1} \cap \mathcal{T}^{\tau}_{J_2} \right) \setminus \Sigma^{\tau 2^{\frac{1}{p}}}_{K-1} \subset \Sigma^{\tau 2^{\frac{1}{p}}}_{K-1} \setminus \Sigma^{\tau 2^{\frac{1}{p}}}_{K-1} = \boldsymbol{\emptyset}$$

Lower bound

Since $\left\{\mathcal{T}_J^\tau \setminus \Sigma_{K-1}^{\tau 2^{\frac{1}{p}}} \ : \ J \in \mathcal{J}(K)\right\}$ are **disjoint**

$$\mathbb{L}^N\big(\Sigma_K^\tau \cap B_q^N(\theta)\big) \geq \sum_{J\in\mathcal{J}(K)} \mathbb{L}^N\Big(\big(\mathcal{T}_J^\tau \setminus \Sigma_{K-1}^{\tau 2^{\frac{1}{p}}}\big) \cap B_q^N(\theta)\Big)$$

Note that $\big(\mathcal{T}_J^\tau \setminus \Sigma_{K-1}^{\tau 2^{\frac{1}{p}}}\big) \cap B_q^N(\theta) \supset \big(\mathcal{T}_J^\tau \cap B_q^N(\theta)\big) \setminus \underbrace{\bigcup_{j\in J}\big(\mathcal{T}_{J\setminus\{j\}}^{\tau 4^{\frac{1}{p}}} \cap \mathcal{T}_J^\tau \cap B_q^N(\theta)\big)}_{\subset\, (\mathcal{T}_J^\tau \cap B_q^N(\theta))}$

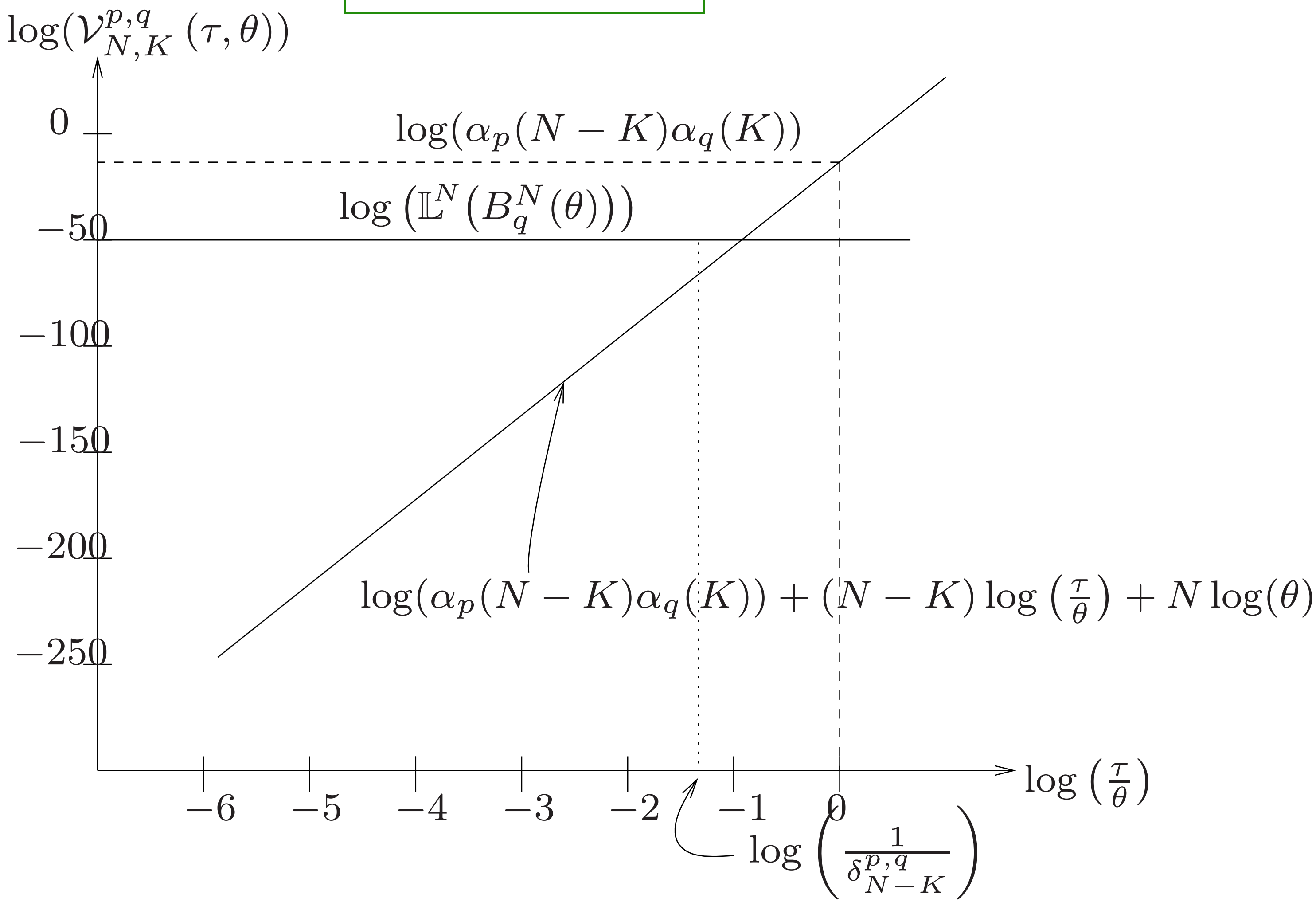
$$\Rightarrow \mathbb{L}^N\big(\Sigma_K^\tau \cap \mathcal{L}_{\|.\|_q}(\theta)\big) \geq \sum_{J\in\mathcal{J}(K)} \underbrace{\mathbb{L}^N\big(\mathcal{T}_J^\tau \cap B_q^N(\theta)\big)}_{\mathcal{V}_{N,K}^{p,q}(\tau,\theta)} - \sum_{J\in\mathcal{J}(K)} \mathbb{L}^N\Big(\bigcup_{j\in J}\big(\mathcal{T}_{J\setminus\{j\}}^{\tau 4^{\frac{1}{p}}} \cap \mathcal{T}_J^\tau \cap B_q^N(\theta)\big)\Big)$$

We find $\mathbb{L}^N\big(\mathcal{T}_J^\tau \cap \mathcal{T}_{J\setminus\{j\}}^{\tau 4^{\frac{1}{p}}} \cap B_q^N(\theta)\big) \leq 2\tau 4^{\frac{1}{p}} \mathcal{V}_{N-1,K-1}^{p,q}(\tau,\theta)$. Hence

Theorem

$$\boldsymbol{\mathbb{L}^N\big(\Sigma_K^\tau \cap B_q^N(\theta)\big) \geq \mathcal{C}_N^K\Big(\mathcal{V}_{N,K}^{p,q}(\tau,\theta) - 2\tau K 4^{\frac{1}{p}} \mathcal{V}_{N-1,K-1}^{p,q}(\tau,\theta)\Big)}$$

6. Simulations

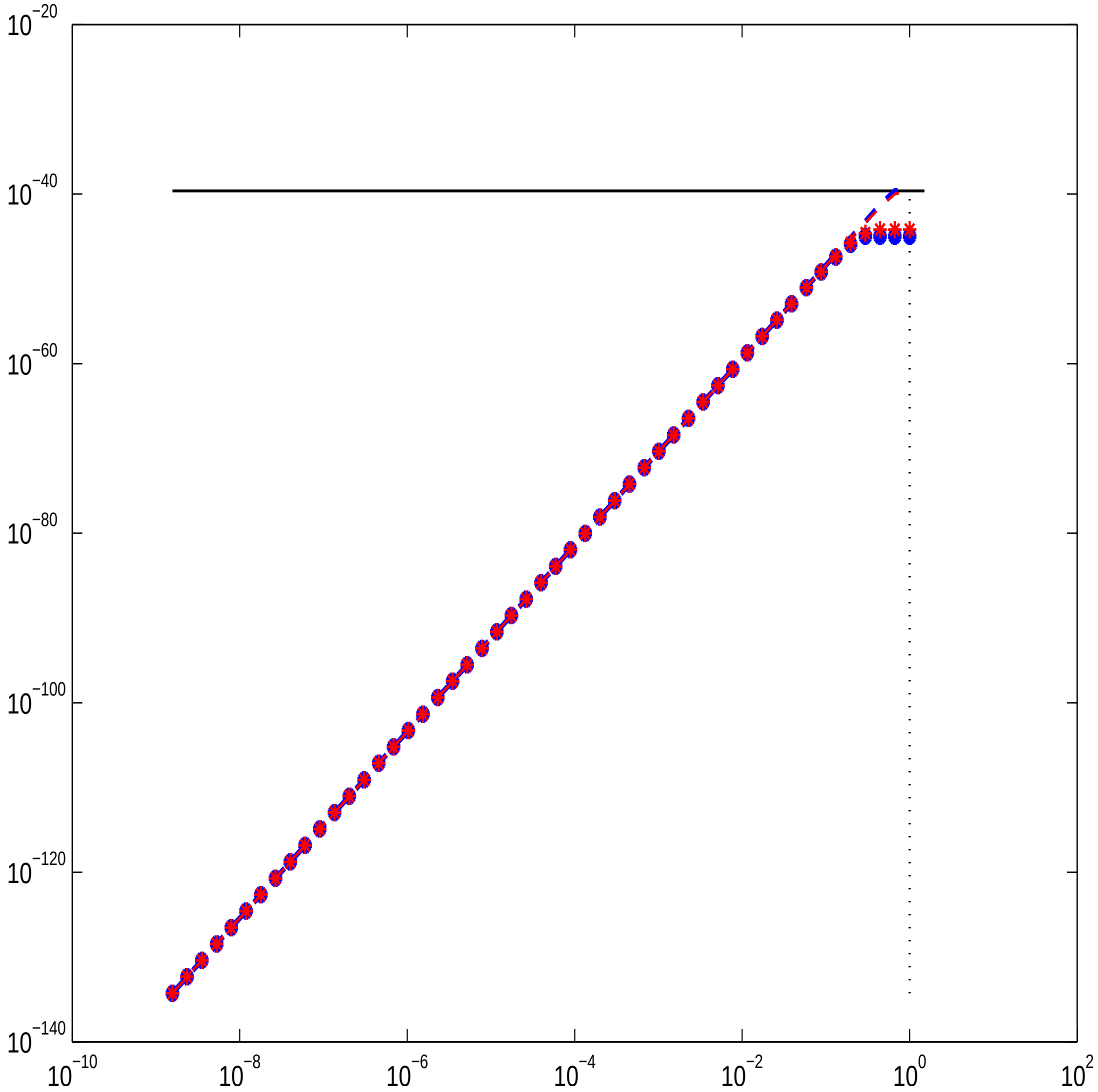


$$\alpha_p(N-K)\alpha_q(K)\left(\tfrac{\tau}{\theta}\right)^{N-K}\theta^N\mathcal{L}^{p,q}_K(\tau,\theta) \le \mathcal{V}^{p,q}_{N,K}(\tau,\theta) \le \alpha_p(N-K)\alpha_q(K)\left(\tfrac{\tau}{\theta}\right)^{N-K}\theta^N\mathcal{U}^{p,q}_K(\tau,\theta)$$

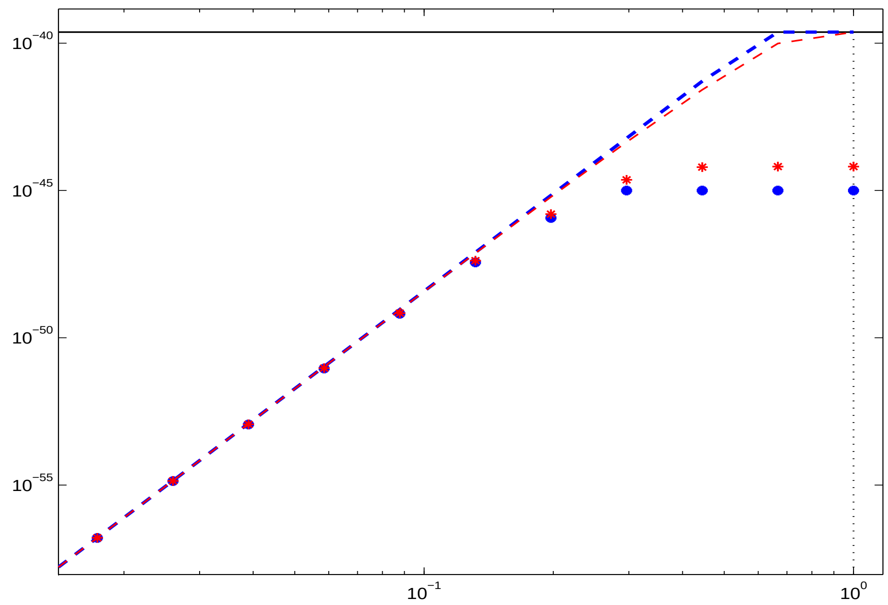
$\mathcal{L}^{p,q}_K(\tau,\theta)$ and $\mathcal{U}^{p,q}_K(\tau,\theta)$ are the limits of *Riemann sums*. Using *Cylinder inclusions* for $\mathcal{T}^\tau_J \cap B^N_q(\theta)$:

$$\theta > \delta^{p,q}_{N-K}\tau > 0 \Rightarrow 0 < \left(1-\left(\delta^{p,q}_{N-K}\frac{\tau}{\theta}\right)^q\right)^{K/q} \le \mathcal{L}^{p,q}_K(\tau,\theta) \le \mathcal{U}^{p,q}_K(\tau,\theta) \quad \text{and} \quad \theta > \tau > 0 \Rightarrow \mathcal{U}^{p,q}_K(\tau,\theta) \le 1$$

Case $p = 1,\ q = 2,\ K = 90,\ N = 100$

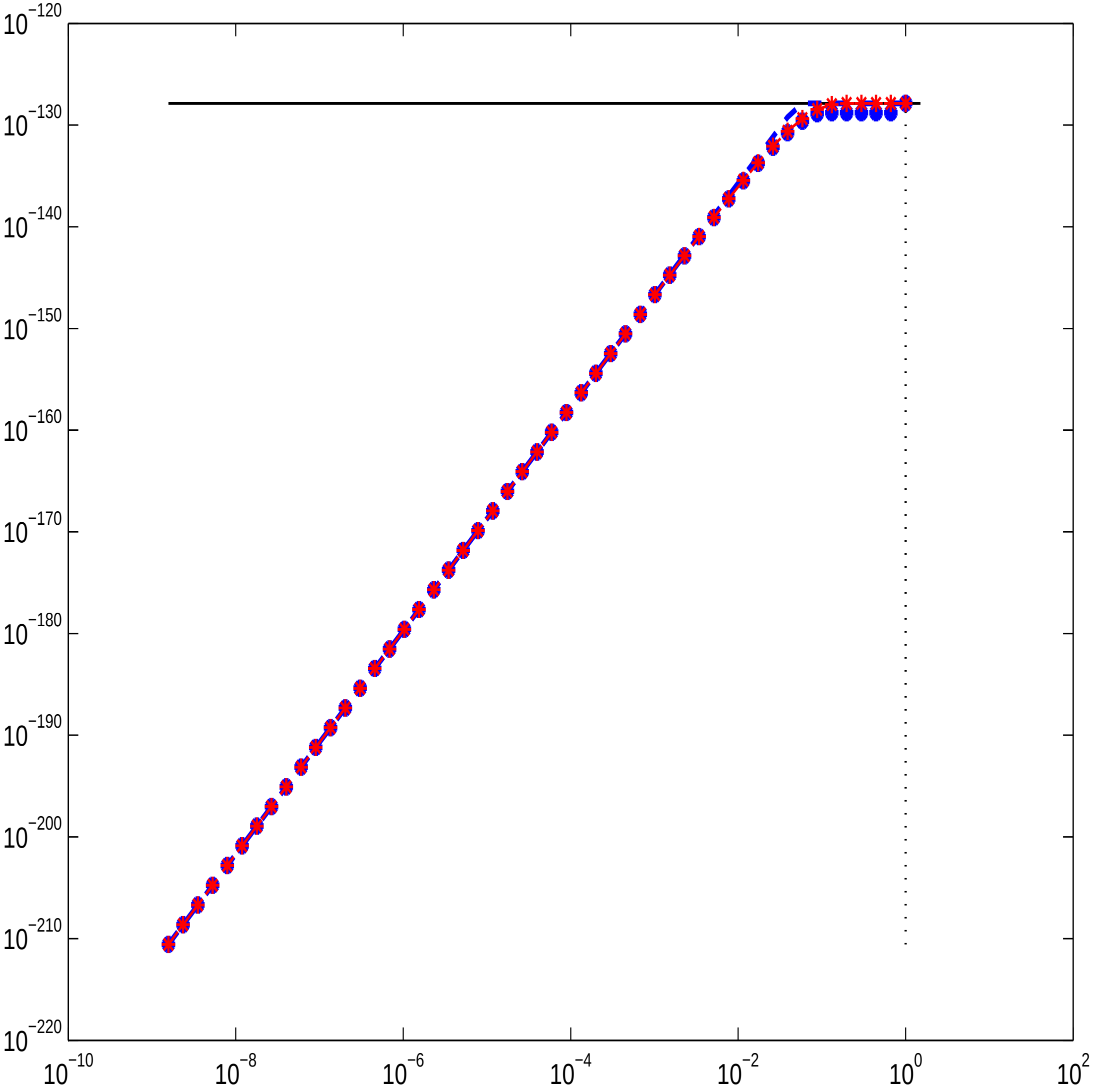


......	Value of τ/θ where the inner cylinder disappears
———	Total mass of the ball
– – – (blue)	Upper bound—Cylinder inclusions
* * (blue)	Lower bound—Cylinder inclusions
– – – (red)	Upper bound—Riemann sum
* * (red)	Lower bound—Riemann sum

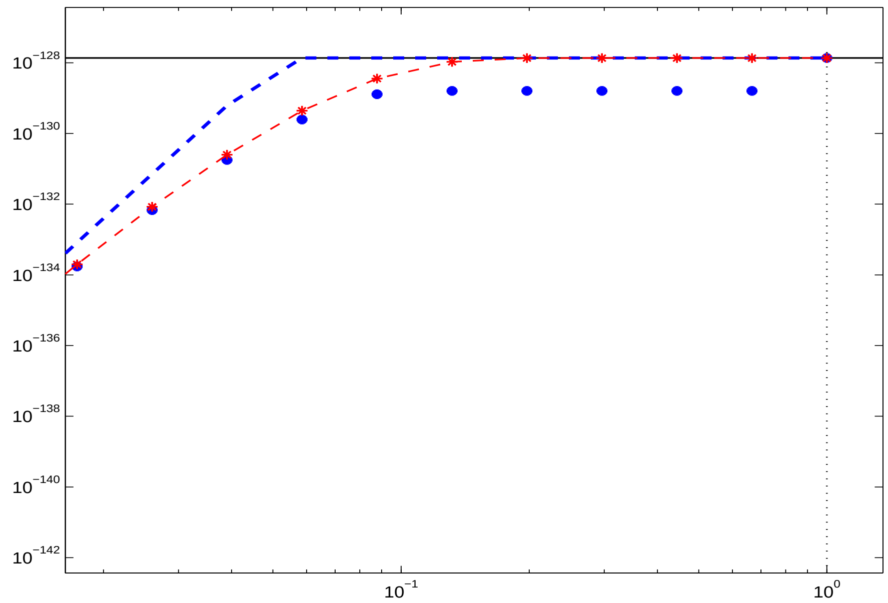


The *Cylinder bounds* are less accurate but close to the *Riemann bounds*. They are accurate for surprisingly large values of $\frac{\tau}{\theta}$.

Case $\boldsymbol{p = 1,\ q = 1,\ K = 90,\ N = 100}$

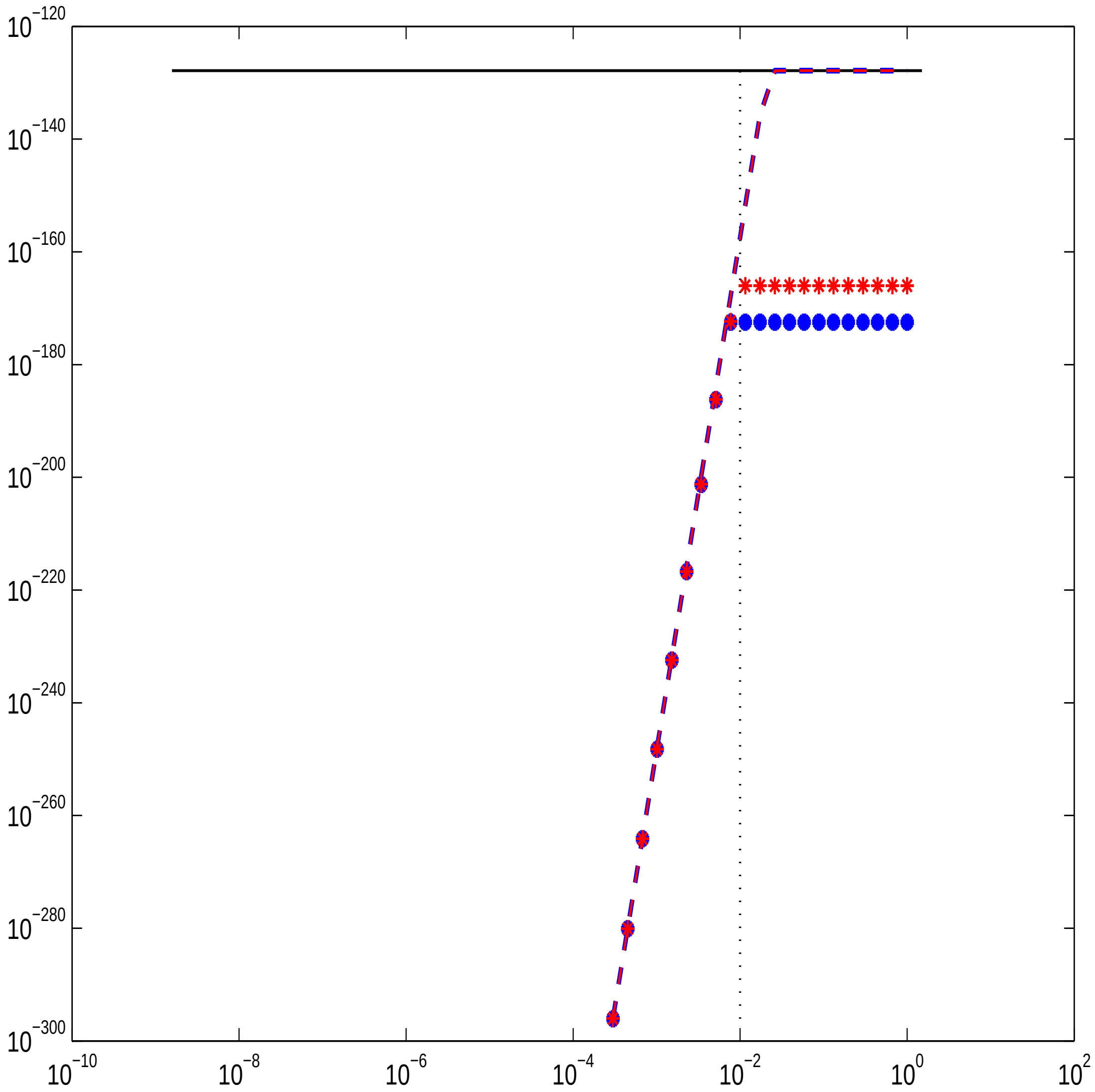


······	Value of τ/θ where the inner cylinder disappears
——	Total mass of the ball
– – – (blue)	Upper bound—Cylinder inclusions
* * (blue)	Lower bound—Cylinder inclusions
– – – (red)	Upper bound—Riemann sum
* * (red)	Lower bound—Riemann sum



The *Cylinder bounds* are less accurate but close to the *Riemann bounds*. The upper and lower bounds computed with Riemann sums are equal.

Case $\boldsymbol{p = \infty,\ q = 1,\ K = 10, N = 100}$



Value of τ/θ where the inner cylinder disappears

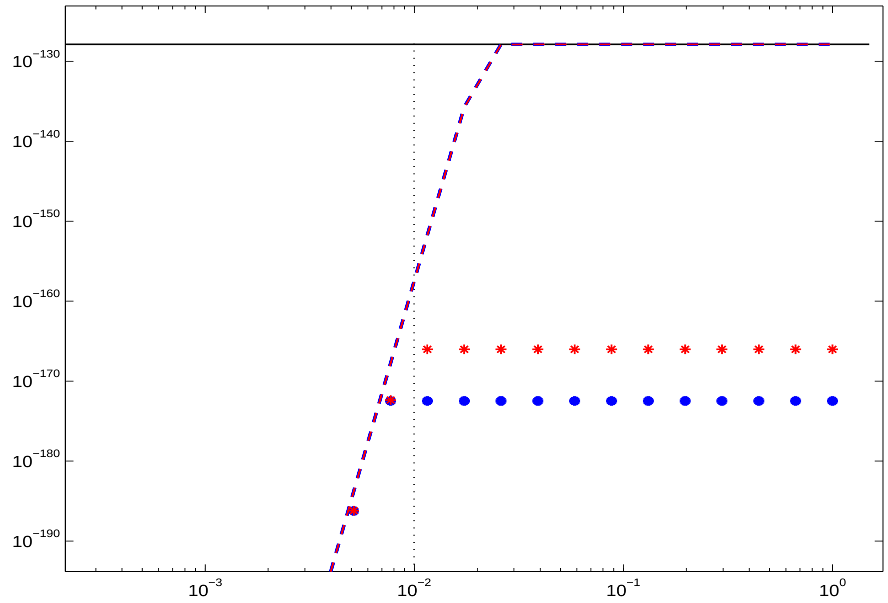
Total mass of the ball

Upper bound—Cylinder inclusions

Lower bound—Cylinder inclusions

Upper bound—Riemann sum

Lower bound—Riemann sum



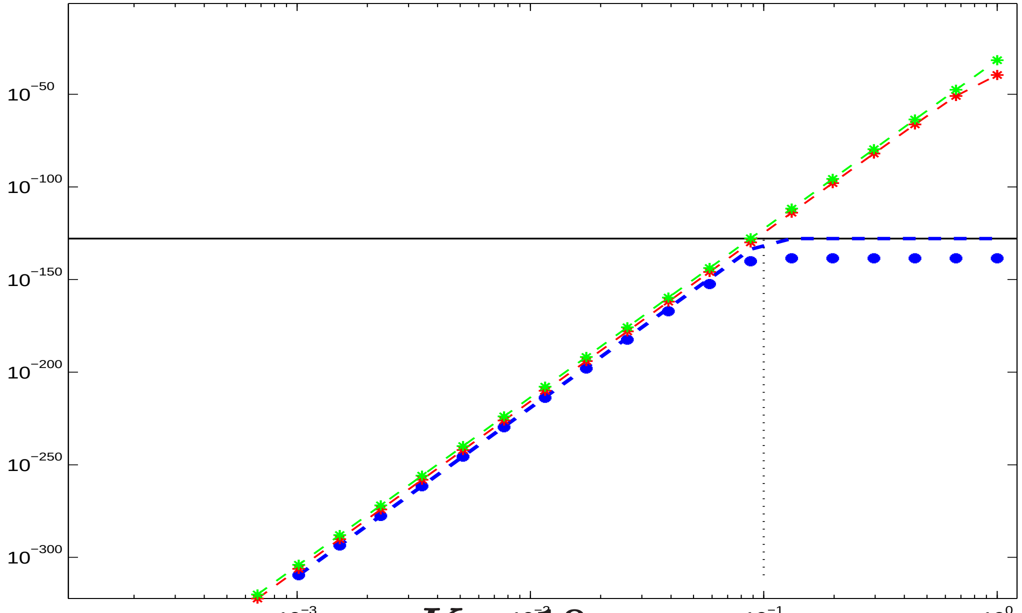
The difference between the *Cylinder* and the *Riemann bounds* is high. This case is less favorable.

Riemann sums for $p = 2$

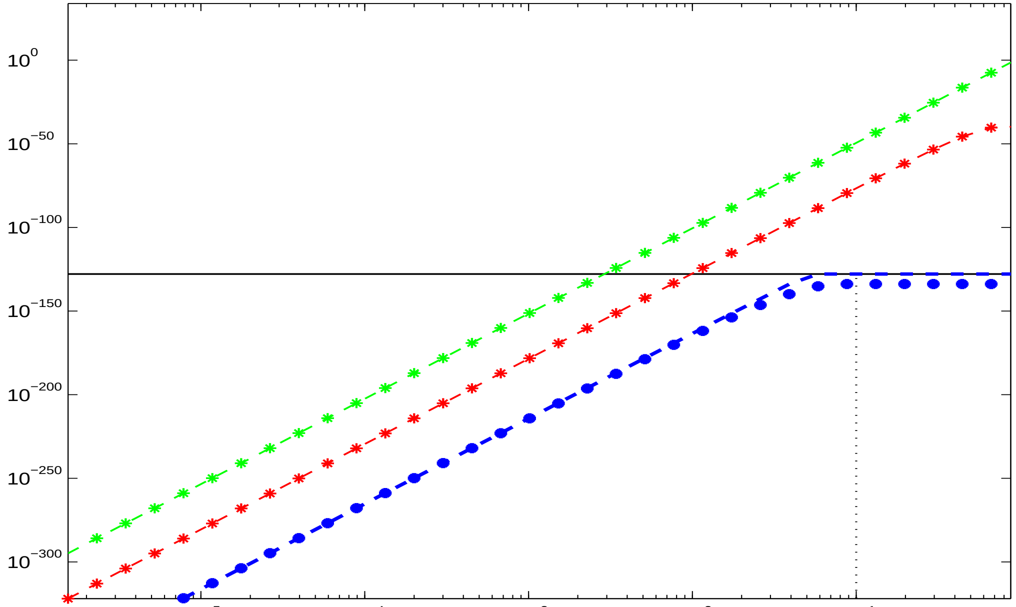
$q = 1$ Blue

$q = 2$ Red

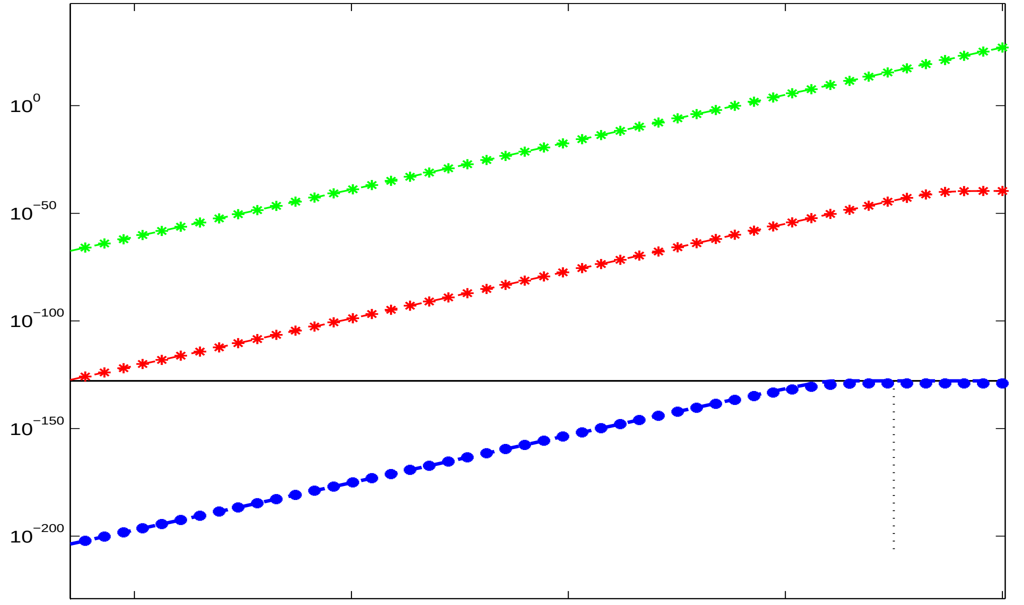
$q = \infty$ Green



$K = 10$

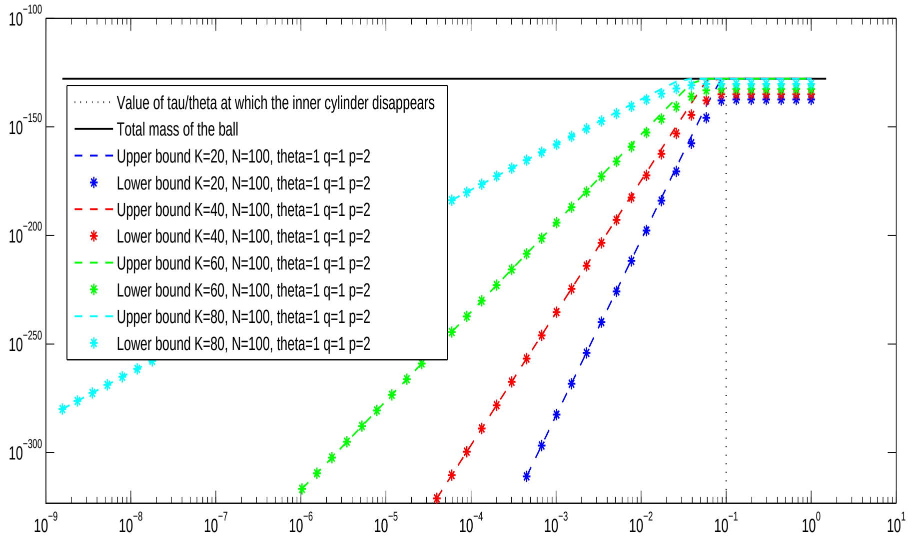


$K = 50$



$K = 90$

Lines are indeed parallel. The difference between curves is due to $\alpha_p(N-K)\alpha_q(K)$.



$$q = 1$$

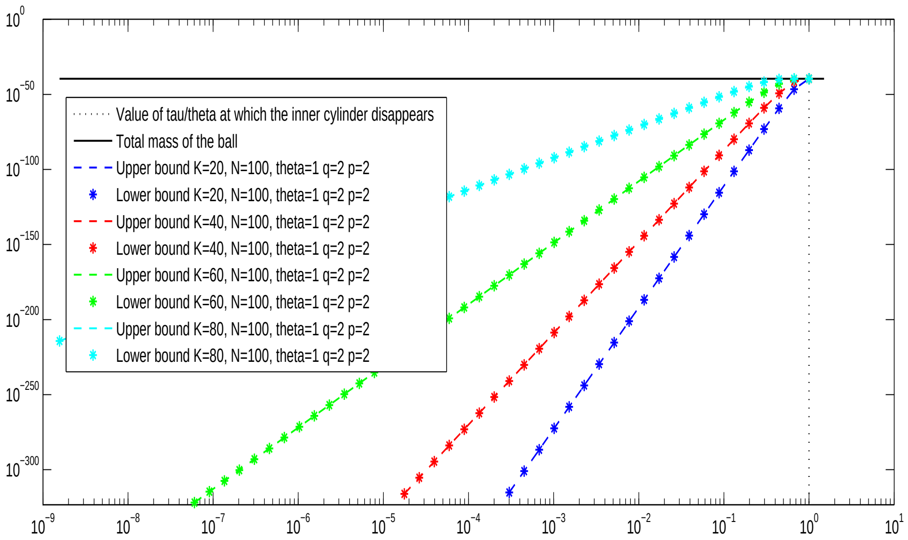
Riemann sums for $p = 2$

$K = 20$ Blue

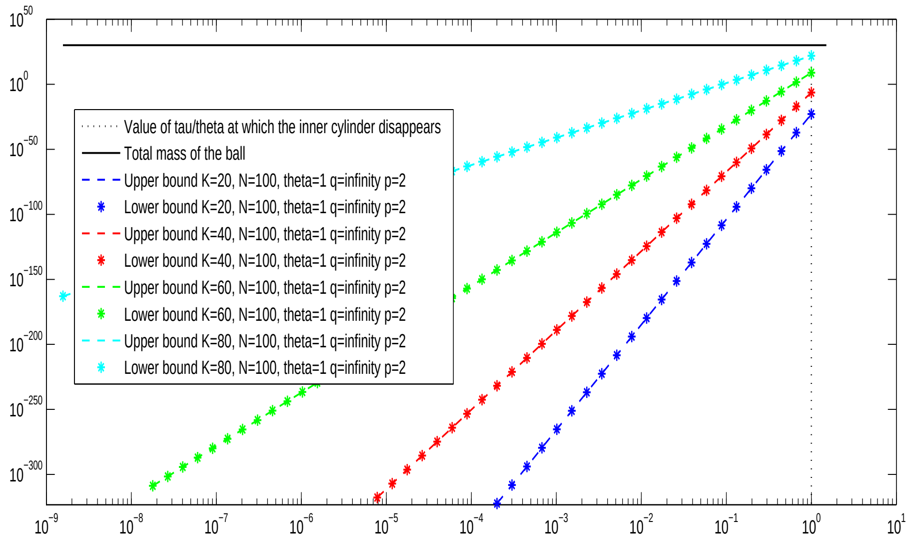
$K = 40$ Red

$K = 60$ Green

$K = 80$ Turquoise



$$q = 2$$



$$q = \infty$$

7. Statistical meaning

Data $d \sim$ Uniform$(B_q^N(\theta)) \Rightarrow$ Bounds$(\mathbb{P}(\mathrm{val}(\mathcal{P}_d) \leq K)) = \dfrac{\text{Bounds}(\mathbb{L}^N(\Sigma_K^\tau \cap B_q^N(\theta)))}{\mathbb{L}^N(D) = \theta^N \alpha_q(N)}$

Theorem

$$(i) \quad \mathbb{P}(\mathrm{val}(\mathcal{P}_d) \leq K) \leq \frac{\mathcal{C}_N^K}{\theta^N \alpha_q(N)} \mathcal{V}_{N,K}^{p,q}(\tau, \theta)$$

$$(ii) \quad \mathbb{P}(\mathrm{val}(\mathcal{P}_d) \leq K) \geq \frac{\mathcal{C}_N^K}{\theta^N \alpha_q(N)} \left(\mathcal{V}_{N,K}^{p,q}(\tau, \theta) - \tau 2K 4^{\frac{1}{p}} \mathcal{V}_{N-1,K-1}^{p,q}(\tau, \theta)\right)$$

Data live in a wide domain (θ is large), tolerance τ is small (for higher precision):

$$\mathcal{V}_{N,K}^{p,q}(\tau, \theta) \sim \alpha_p(N-K)\alpha_q(K)\left(\tfrac{\tau}{\theta}\right)^{N-K} \theta^N \quad \text{as } \frac{\tau}{\theta} \to 0$$

Theorem [Asymptotics]

$$\mathbb{P}(\mathrm{val}(\mathcal{P}_d) \leq K) \sim \mathcal{C}_N^K \alpha_p(N-K) \frac{\alpha_q(K)}{\alpha_q(N)} \left(\frac{\tau}{\theta}\right)^{N-K} \quad \text{as } \frac{\tau}{\theta} \to 0$$

$$\frac{\mathbb{P}\left(\mathrm{val}(\mathcal{P}_d^M) \le K\right)}{\mathbb{P}\left(\mathrm{val}(\mathcal{P}_d) \le K\right)}$$

$$\frac{\mathcal{C}_M^K \alpha_2(K)\alpha_1(N)}{\mathcal{C}_N^K \alpha_1(K)\alpha_2(N)\beta^{N-K}}$$

τ/θ fixed, $\beta = 1$

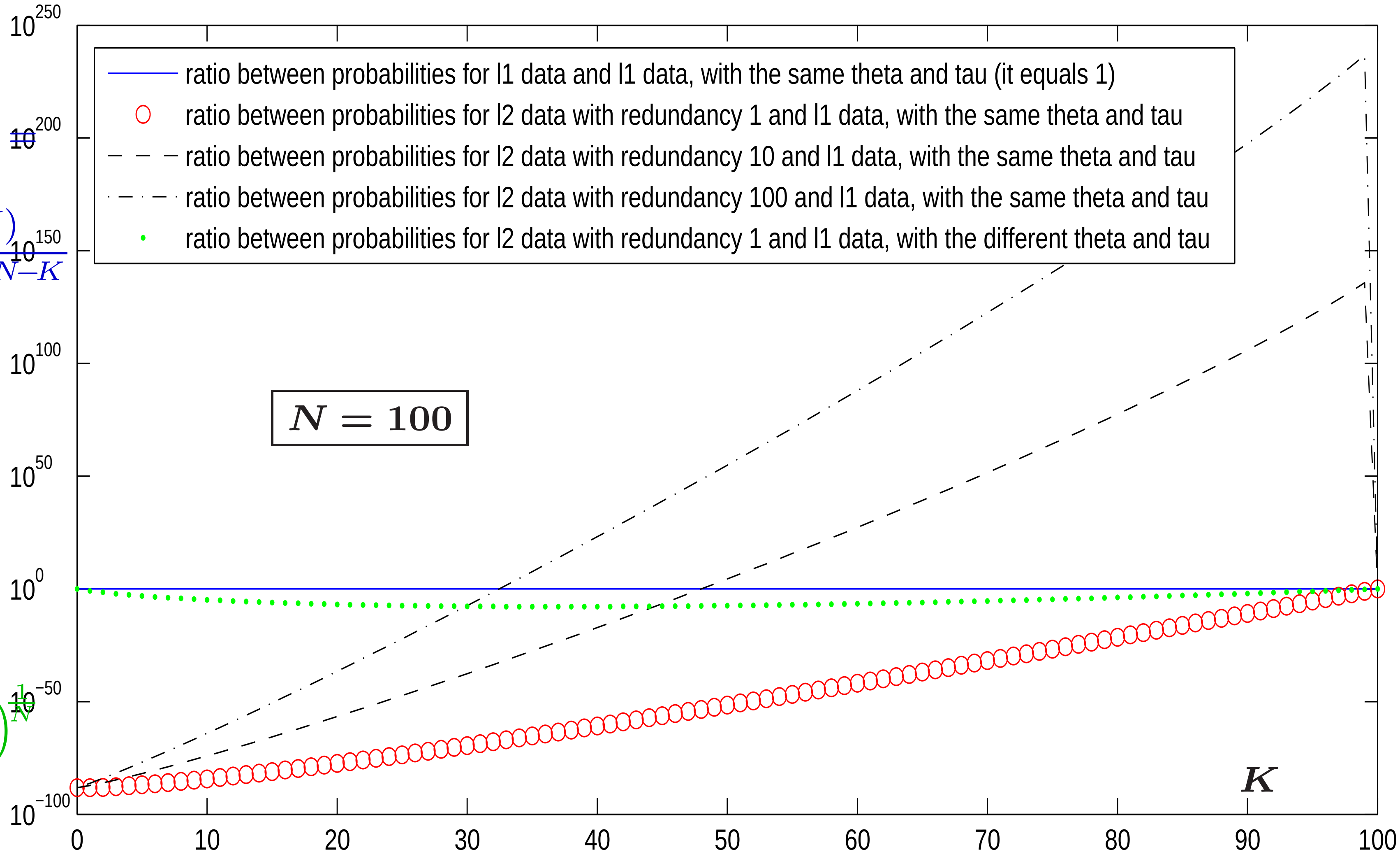
Level $\mathbf{1}$: ——

$M = N$ ○ ○ ○ ○

$\frac{M}{N} = 10$ – – –

$\frac{M}{N} = 100$ - . - . .

$M{=}N, \beta{=}\left(\frac{\alpha_1(N)}{\alpha_2(N)}\right)^{\frac{1}{N}}$ . . . . . .



$$(\mathcal{P}_d) \quad \min \|u\|_0 : \quad \|u - d\|_2 \le \tau, \ d \sim \text{uniform on } B_1^N(\theta)$$

$$(\mathcal{P}_d^M) \quad \min \|u\|_0 : \quad \|Au - d\|_2 \le \tau, \ A \in \mathrm{R}^{N\times M}, \mathrm{rank}(A) = N, \ d \sim \text{uniform on } B_2^N(\beta\theta)$$

$$\mathbb{P}\left(\mathrm{val}(\mathcal{P}_d^M) \le K\right) = \mathcal{C}_M^K \frac{\alpha_2(N-K)\alpha_2(K)}{\alpha_2(N)} \left(\frac{\tau}{\theta}\right)^{N-K} \frac{1}{\beta^{N-K}}$$

9. Conclusions and open questions

- APA methodology in the case of an ℓ_0 objective and an orthonormal basis leads to finer bounds and explicit formulae.
- Getting a higher $\mathbb{P}\,(\mathrm{val}(\mathcal{P}_d) \le K)$ for K small needs we have p and q such that $\alpha_p(N-K)\frac{\alpha_q(K)}{\alpha_q(N)}$ is large for K small, and vice-versa.
- Precise evaluation of subsets of the form $\mathbb{L}^N(\mathcal{T}^{\tau}_{J_1} \cap \mathcal{T}^{\tau}_{J_2} \cap B^N_q(\theta))$ is an open problem.
- Computing $\mathbb{L}^N(\Sigma^{\tau}_K \cap B^N_q(\theta))$ using Riemann sums is better than Cylinder inclusions.
- If $q = \infty$, measures are exact; however, this case has a limited practical interest.
- Cases $q \le p$ are more important in practice. Our bounds need further refinement.

Thanks for your kind attention!

For more details see: http://www.cmla.ens-cachan.fr/∼nikolova/