Inverse modeling in inverse problems using optimization

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Beijing, 14-17 July, 2014
object $u_o$ $\rightarrow$ capture energy $\rightarrow$ sampling quantization $\rightarrow$ processing

$\begin{pmatrix}
\text{scene} \\
\text{body} \\
\text{earth}
\end{pmatrix}$ $\rightarrow$ $\begin{pmatrix}
\text{reflected} \\
\text{or} \\
\text{emitted}
\end{pmatrix}$ $\rightarrow$ $\begin{pmatrix}
\text{signal} \\
\text{or} \\
\text{image}
\end{pmatrix}$

Mathematical model: $v = \text{Transform}(u_o) \bullet (\text{Perturbations})$

Some transforms: loss of pixels, blur, FT, Radon T., frame T. ($\cdots$)

Processing tasks: $\begin{cases}
\hat{u} = \text{recover}(u_o) \\
\hat{u} = \text{objects of interest}(u_o)
\end{cases}$ ($\cdots$)

Mathematical tools: PDEs, Statistics, Functional anal., Matrix anal., ($\cdots$)
Example due to R.S. Wilson

\( u_o \) (unknown—signal, picture, density map) \( v \) (data, degraded) = \text{Transform}(u_o) \cdot n \) (noise)

An ill-posed inverse problem

\[ u_o = [1 \ 1 \ 1 \ 1]^T \]

Transform:

\[
A = \begin{bmatrix}
10 & 7 & 8 & 7 \\
7 & 5 & 6 & 5 \\
8 & 6 & 10 & 9 \\
7 & 5 & 9 & 10 \\
\end{bmatrix}
\]

\( \text{rank}(A) = 4 \)

- no noise: \( v = Au_o = [32 \ 23 \ 33 \ 31]^T \Rightarrow \hat{u} = A^{-1}v = u_o \)

- with noise: \( v = Au_o + n = [32.1 \ 22.9 \ 33.1 \ 30.9]^T \)

Least-squares solution: \( \hat{u} = \arg \min_{u \in \mathbb{R}^4} \|Au - v\|^2 \) = \( A^{-1}v \)

\[ \Rightarrow \hat{u} = [9.2 \ -12.6 \ 4.5 \ -1.1]^T \]

Tikhonov regularization: \( \hat{u} = \arg \min_{u \in \mathbb{R}^4} \mathcal{F}_v(u) \)

\[
\mathcal{F}_v(u) \overset{\text{def}}{=} \|Au - v\|^2 + \beta \sum_{i=1}^{3} (u[i+1] - u[i])^2
\]

\( \beta = 1 \Rightarrow \hat{u} = [1 \ 1.01 \ 1.02 \ 0.98]^T \)
Image/signal processing tasks often require to solve **ill-posed inverse problems**

Out-of-focus picture: \( v = a * u_o + \text{noise} = A u_o + \text{noise} \)

\( A \) is ill-conditioned \( \equiv \) (nearly) noninvertible

Least-squares solution: \( \hat{u} = \arg\min_u \{ \|Au - v\|^2 \} \)

Tikhonov regularization: \( \hat{u} := \arg\min_u \{ \|Au - v\|^2 + \beta \sum_i ||G_i u||^2 \} \) for \( \{G_i\} \approx \nabla, \beta > 0 \)
Formulate your problem as the minimization (maximization) of a functional (an energy) whose solution is the sought after signal/image.
Goal of this tutorial: How to choose your energy $\mathcal{F}_v$?

Approach: Salient features of the minimizers of classes of energies $\mathcal{F}_v$

Outline

1. Energy minimization methods (p. 7)
2. Regularity results (p. 17)
3. Non-smooth regularization – minimizers are sparse in a given subspace (p. 26)
4. Non-smooth data-fidelity – minimizers fit exactly some data entries (p. 35)
5. Comparison with Fully Smooth Energies (p. 51)
6. Non-convex regularization – edges are sharp (p. 54)
7. Nonsmooth data-fidelity and regularization – peculiar features (p. 62)
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10. Some References (p. 103)
1. Energy minimization methods

\[ u_o \text{ (unknown)} \quad v \text{ (data)} = \text{Transform}(u_o) \bullet \text{(Perturbations)} \]

\[ \hat{u} \text{ solution} \]

\[ \begin{align*}
\hat{u} \nearrow & \text{ close to data production model } \Psi(u, v) \text{ (data-fidelity)} \\
\hat{u} \searrow & \text{ coherent with priors and desiderata } \Phi(u) \text{ (prior – functional, constraint )}
\end{align*} \]

Combining models:

\[ \hat{u} := \arg \min_{u \in \Omega} F_v(u) \quad (P) \]

\[ F_v(u) := \Psi(u, v) + \beta \Phi(u), \quad \beta > 0 \]

\[ \boxed{\text{How to choose } (P) \text{ to get a good } \hat{u} ?} \]

Applications: Denoising, Segmentation, Deblurring, Tomography, Seismic imaging, Zoom, Superresolution, Compression, Learning, Motion estimation, Pattern recognition (⋯)

The \( m \times n \) image \( u \) is stored in a \( p = mn \)-length vector, \( u \in \mathbb{R}^p \), data \( v \in \mathbb{R}^q \).
\( \Psi \) usually models the production of data \( v \)  \( \Rightarrow \)  \( \Psi = - \log( \text{Likelihood } (v|u) ) \)

\[
v = Au_0 + n \text{ for } n \text{ white Gaussian noise} \Rightarrow \Psi(u, v) \propto \|Au - v\|^2_2
\]

The information on \( u \) we have is implicitly contained in \( \Psi \). It is scarcely enough.
A good prior \( \Phi \) is needed to solve our task.

**\( \Phi \) model for the unknown \( u \)  (statistics, smoothness, edges, textures, expected features)***

- Bayesian approach
- Variational approach

Both approaches lead to similar energies

Prior via regularization term  \( \Phi(u) = \sum_i \varphi(\|G_iu\|) \)

\( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) potential function (PF)

\( \{G_i\} \) — linear operators.  Examples: \( \text{Id}, \nabla, \nabla^2, \nabla \tilde{W} \) for \( \tilde{W} \) left inverse of a frame if \( u = W(\text{image}) \)
Bayes: $U$, $V$ random variables, Likelihood $f_{V|U}(v|u)$, Prior $f_U(u) \propto \exp\{-\lambda \Phi(u)\}$

Maximum a Posteriori (MAP) yields the most likely solution $\hat{u}$ given the data $V = v$:

$$\hat{u} = \arg \max_u f_{U|V}(u|v) = \arg \min_u \left( -\ln f_{V|U}(v|u) - \ln f_U(u) \right)$$

$$= \arg \min_u \left( \Psi(u, v) + \beta \Phi(u) \right) = \arg \min_u F_v(u)$$

MAP is a very usual way to combine models on data-acquisition and priors

Realist models for data-acquisition $f_{V|U}$ and prior $f_U$ is still an open question

Are you satisfied with the solution?
• **Minimizer approach**  (the core of our tutorials)

  – Analyze the main properties exhibited by the (local) minimizers $\hat{u}$ of $\mathcal{F}_v$ as a function of the shape of $\mathcal{F}_v$

  Strong results.

  **Rigorous tools for modelling**

  – Conceive $\mathcal{F}_v$ so that the properties of $\hat{u}$ satisfy your requirements.

  (a “chicken and egg” problem?)

  “There is nothing quite as practical as a good theory.”  Kurt Lewin
Illustration: the role of the smoothness of $\mathcal{F}_v$

$\mathcal{F}_v(u) = \sum_{i=1}^{p} (u_i - v_i)^2 + \beta \sum_{i=1}^{p-1} |u_i - u_{i+1}|$

- smooth non-smooth

$\mathcal{F}_v(u) = \sum_{i=1}^{p} |u_i - v_i| + \beta \sum_{i=1}^{p-1} (u_i - u_{i+1})^2$

- non-smooth smooth

$\mathcal{F}_v(u) = \sum_{i=1}^{p} |u_i - v_i| + \beta \sum_{i=1}^{p-1} |u_i - u_{i+1}|$

- non-smooth non-smooth

$\mathcal{F}_v(u) = \sum_{i=1}^{p} (u_i - v_i)^2 + \beta \sum_{i=1}^{p-1} (u_i - u_{i+1})^2$

- smooth smooth

We shall explain why and how to use
Some energy functions

Regularization [Tikhonov, Arsenin 77]: \( \mathcal{F}_v(u) = \|Au - v\|^2 + \beta\|Gu\|^2, \ G = I \) or \( G \approx \nabla \)

Focus on edges, contours, segmentation, labeling

Statistical framework

Potts model [Potts 52] (\( \ell_0 \) semi-norm applied to differences):

\[
\mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{i,j} \phi(u[i] - u[j]) \phi(t) := \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } t \neq 0
\end{cases}
\]

Line process in Markov random field priors [Geman, Geman 84]: \((\hat{u}, \hat{l}) = \arg \min_{u, l} \mathcal{F}_v(u, l)\)

\[
\mathcal{F}_v(u, l) = \Psi(u, v) + \beta \sum_i \left( \sum_{j \in \mathcal{N}_i} \varphi(u[i] - u[j])(1 - l_{i,j}) + \sum_{(k,n) \in \mathcal{N}_{i,j}} V(l_{i,j}, l_{k,n}) \right)
\]

[\( l_{i,j} = 0 \iff \text{no edge} \], \[ l_{i,j} = 1 \iff \text{edge between } i \text{ and } j \], \( \varphi(t) = 1 \)

“We make an analogy between images and statistical mechanics systems. Pixel gray levels and the presence and orientation of edges are viewed as states of atoms or molecules in a lattice-like physical system. The assignment of an energy function in the physical system determines its Gibbs distribution. Because of the Gibbs distribution, Markov random field (MRF) equivalence, this assignment also determines an MRF image model.” [S. Geman, D. Geman 84]
PDE's framework

M.-S. functional [Mumford, Shah 89]:
\[ F_v(u, L) = \int_{\Omega} (u - v)^2 \, dx + \beta \left( \int_{\Omega \setminus L} \| \nabla u \|^2 \, dx + \alpha |L| \right) \]

discrete version: \[ \Phi(u) = \sum_i \varphi(\| G_i u \|), \quad \varphi(t) = \min\{t^2, \alpha\}, \quad \{G_i\} \approx \nabla \]

Total Variation (TV) [Rudin, Osher, Fatemi 92]:
\[ F_v(u) = \| u - v \|_2^2 + \beta \, \text{TV}(u) \]
\[ \text{TV}(u) = \int \| \nabla u \|_2 \, dx \approx \sum_i \| G_i u \|_2 \]

Various edge-preserving functions \( \varphi \) to define \( \Phi \)
\[ \varphi \text{ is edge-preserving if } \lim_{t \to \infty} \frac{\varphi'(t)}{t} = 0 \]
[Charbonnier, Blanc-Féraud, Aubert, Barlaud 97 ...]

Minimizer approach

\( \ell_1 \) — Data fidelity [Nikolova 02]:
\[ F_v(u) = \| Au - v \|_1 + \beta \Phi(u) \]

\( L_1 - \text{TV} \) model [T. Chan, Esedoglu 05]:
\[ F_v(u) = \| u - v \|_1 + \beta \text{TV}(u) \]

CPU time! Computers ↑↑
\[ \mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum_i \varphi((\nabla u)[i]) \]

\( \varphi(t) = |t|^{\alpha \in (1,2)} \)
\( \varphi(t) = |t| \)

\( \varphi \) smooth at 0
\( \varphi \) nonsmooth at 0

\( \varphi(t) = \alpha t^2 / (1 + \alpha t^2) \)
\( \varphi(t) = \alpha |t| / (1 + \alpha |t|) \)
\( \varphi(t) = \min\{\alpha t^2, 1\} \)
\( \varphi(t) = 1 - \mathbb{1}_{t=0} \)

convex
nonconvex
nonconvex
nonconvex
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2 Regularity Results

Optimization problems

\( F(v) \) nonconvex

\( F(v) \) convex non coercive \( \Omega = \mathbb{R} \)

\( F(v) \) convex non coercive \( \Omega \) compact

\( F(v) \) strictly convex, \( \Omega \) nonconvex

\( F(v) \) non strictly convex

\( F(v) \) strictly convex on \( \mathbb{R} \)
\[ \mathcal{F}_v : \Omega \to \mathbb{R} \quad \Omega \subset \mathbb{R}^p \]

- Set of optimal solutions \( \widehat{U} = \{ \hat{u} \in \Omega : \mathcal{F}_v(\hat{u}) \leq \mathcal{F}_v(u) \quad \forall u \in \Omega \} \)
  \( \widehat{U} = \{ u \} \) if \( \mathcal{F}_v \) strictly convex
  \( \widehat{U} \neq \emptyset \) if \( \mathcal{F}_v \) coercive or if \( \mathcal{F}_v \) continuous and \( \Omega \) compact
  Otherwise – check
  (e.g. see if \( \mathcal{F}_v \) is asymptotically level stable [Auslender, Teboulle 03])

- Nonconvex problems:
  Algorithms may get trapped in local minima
  A “good” local minimizer can be satisfying
  Global optimization – difficult, but progress, e.g. [Robini, Reissman JGO 13]
  Convex relaxation methods, see, e.g., [Yuan, Bae, Tai CVPR 10]

- Attention to numerical errors
Definition: $U : O \to \mathbb{R}^p$, $O \subset \mathbb{R}^q$ open, is a (local) minimizer function for

$$\mathcal{F}_O := \{ F_v : v \in O \}$$

if $F_v$ has a strict (local) minimum at $U(v)$, $\forall v \in O$.

Minimizer functions – an useful tool to analyze the properties of minimizers...

Each blue curve curve: $u \to F_v(u)$ for $v \in \{0, 2, \cdots \}$

**Question 1** What these plots reveal about the local / global minimizer functions?
\[ \mathcal{F}_v(u) = \|Au - v\|_2^2 + \beta \Phi(u) \]

\[ \Phi(u) = \sum_i \varphi(\|G_i u\|_2) \]

\[ u \in \mathbb{R}^p \]

\[ v \in \mathbb{R}^q \]

\[ \{G_i\} \text{ linear operators } \mathbb{R}^p \to \mathbb{R}^s, \ s \geq 1 \]

\[ \varphi'(0^+) > 0 \implies \Phi \text{ is nonsmooth on } \bigcup_i \{u : G_i u = 0\} \]

Systematically: \( \ker A \cap \ker G = \{0\} \)

\[ G := \begin{bmatrix} G_1 \\ G_2 \\ \vdots \end{bmatrix} \]

Recall:

\[ \mathcal{F}_v \text{ has a (local) minimum at } \hat{u} \implies \delta \mathcal{F}_v(\hat{u})(d) = \lim_{t \downarrow 0} \frac{\mathcal{F}_v(u + td) - \mathcal{F}_v(u)}{t} \geq 0, \forall d \in \mathbb{R}^p \]

\[ \mathcal{F}_v \text{ nonconvex } \implies \text{there may be many local minima} \]
• $N = \{(s, t) : t = \pm \arctan(s)\}$

• $N$ is closed in $\mathbb{R}^2$ and its Lebesgue measure in $\mathbb{R}^2$ is $L^2(N) = 0$

• $(x, y) = \text{random } \mathbb{R}^2$

**Question 2** What is the chance that $(x, y) \in N$?
Stability of the minimizers of $\mathcal{F}_v$ [Durand & Nikolova 06]

Assumptions: $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous and $C^{m \geq 2}$ on $\mathbb{R}_+ \setminus \{\theta_1, \cdots \theta_n\}$, edge-preserving, possibly non-convex and $\text{rank}(A) = p$

A. LOCAL MINIMIZERS

(knowing local minimizers is important)

There is a closed $N \subset \mathbb{R}^q$ with Lebesgue measure $L^q(N) = 0$ such that $\forall v \in \mathbb{R}^q \setminus N$, every (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ is given by $\hat{u} = U(v)$ where $U$ is a $C^{m-1}$ (local) minimizer function.

Question 3 For $v \in \mathbb{R}^q \setminus N$, compare $U(v)$ and $U(v + \epsilon)$ where $\epsilon \in \mathbb{R}^q$ is small enough.

B. GLOBAL MINIMIZERS

- $\exists \hat{N} \subset \mathbb{R}^q$ with $L^q(\hat{N}) = 0$ and $\text{Int}(\mathbb{R}^q \setminus \hat{N})$ dense in $\mathbb{R}^q$ such that $\forall v \in \mathbb{R}^q \setminus \hat{N}$, $\mathcal{F}_v$ has a unique global minimizer.

- There is an open subset of $\mathbb{R}^q \setminus \hat{N}$, dense in $\mathbb{R}^q$, where the global minimizer function $\hat{U}$ is $C^{m-1}$-continuous.

Question 4 What is the chance that $v \in \hat{N}$? What can happen if $v \in \hat{N}$?
Nonasymptotic bounds on minimizers [Nikolova 07]

Classical bounds for $\beta \downarrow 0$ or $\beta \uparrow \infty$

Assumption: $\varphi$ is piecewise $C^1$

- $\varphi$ is strictly increasing or $\text{rank}(A) = p$
  \[ \hat{u} \text{ is a (local) minimizer of } F_v \Rightarrow \|A\hat{u}\| \leq \|v\| \]

- $\|\varphi'\|_{\infty} = \text{constant}$ (\(\varphi\) is edge-preserving) and $\text{rank}(A) = q \leq p$
  \[ \hat{u} \text{ is a (local) minimizer of } F_v \Rightarrow \|v - A\hat{u}\|_{\infty} \leq \frac{\beta}{2} \|\varphi'\|_{\infty} \|(AA^*)^{-1}A\|_{\infty} \|G\|_1 \]

$\|\varphi'\|_{\infty} = 1$, $A = \text{Id}$ and $G$ — 1st order differences:
\[
\begin{align*}
\text{signal} & \Rightarrow \|v - \hat{u}\|_{\infty} \leq \beta \\
\text{image} & \Rightarrow \|v - \hat{u}\|_{\infty} \leq 2\beta
\end{align*}
\]

Question 5 If $v = u_o + n$ for $n$ Gaussian noise, is it possible to clean $v$ from this noise by minimizing $F_v$? (See $\Psi$ on p. 8.)
Non-Smooth Energies, Side Derivatives, Subdifferential

Rademacher’s theorem: If $\mathcal{F}_v : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous, then $\mathcal{F}_v$ is differentiable (in the usual sense) almost everywhere in $\mathbb{R}^p$.

A kink is a point $u$ where $\nabla \mathcal{F}_v(u)$ is not defined (in the usual sense).

Example: $\mathcal{F}_v(u) = \frac{1}{2} (u - v)^2 + \beta |u|$ for $\beta = 1 > 0$ and $u, v \in \mathbb{R}$

$$\hat{u} = \begin{cases} 
  v + \beta & \text{if } v < -\beta \\
  0 & \text{if } |v| \leq \beta \\
  v - \beta & \text{if } v > \beta 
\end{cases}$$

**Question 6** Comment the minimizers on the 1st row. What is drawn on the 2nd row?
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3 Minimizers under Non-Smooth Regularization

\[ \mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|), \quad \Psi \in C^{m>2}, \quad \varphi \in C^m(\mathbb{R}^*_+), \quad 0 < \varphi'(0^+) \leq \infty \]

\[ \varphi(t) \quad t^\alpha, \quad \alpha \in (0, 1) \quad \frac{\alpha t}{\alpha t + 1} \quad \ln(\alpha t + 1) \quad 1 - \alpha^t \quad \alpha \in (0, 1) \quad (\cdots), \quad \alpha > 0 \]

\varphi(t) = t \text{ and } G_i u \approx (\nabla u)_i \quad \Rightarrow \quad \Phi(u) = TV(u) \quad (\text{total variation}) \quad [\text{Rudin, Osher, Fatemi 92}]
General case  \( F_v(u) = \Psi(u, v) + \beta \sum_{i=1}^{r} \varphi(||G_i u||) \)  \( \Psi \in C^{m \geq 2}, \varphi'(0^+) > 0 \)  \cite{Nikolova 97,00}

Let \( \hat{u} \) be a (local) minimizer of \( F_v \). Set \( \hat{h} := \{ i : G_i \hat{u} = 0 \} \)
Then \( \exists \ O \subset \mathbb{R}^q \) open, \( \exists \ U \in C^{m-1} \) (local) minimizer function so that

\[
v' \in O, \ \hat{u}' = U(v') \implies G_i \hat{u}' = 0, \ \forall i \in \hat{h}
\]

\( \hat{h} \subset \{1, .., r\} \) \( O_{\hat{h}} := \{ v \in \mathbb{R}^q : G_i U(v) = 0, \ \forall i \in \hat{h} \} \implies \mathbb{L}^q(O_{\hat{h}}) > 0 \)

Data \( v \) yield (local) minimizers \( \hat{u} \) of \( F_v \) such that
\( G_i \hat{u} = 0 \) for a set of indexes \( \hat{h} \)

\( G_i = \nabla_i \implies \hat{u}[i] = \hat{u}[j] \) for many neighbors \( (i, j) \) (the "stair-casing" effect)

\( G_i u = u[i] \implies \) many samples \( \hat{u}[i] = 0 \) – highly used in Compressed Sensing

**Question 7** What happens if \( \{G_i\} \) yield second-order differences?

Property fails if \( F_v \) is smooth, except for \( v \in N \) where \( N \) is closed and \( \mathbb{L}^q(N) = 0 \).
\[ \mathcal{F}_v(u) = \| u - v \|^2 \]
\[ + \beta \sum \varphi(|u[i] - u[i-1]|) \]

\[ \varphi(t) = \sqrt{\alpha + t^2}, \quad \varphi'(0) = 0 \quad \text{(smooth at 0)} \]
\[ \varphi(t) = (t + \alpha \text{sign}(t))^2, \quad \varphi'(0^+) = 2\alpha \]

\[ \varphi(t) = |t|, \quad \varphi'(0^+) = 1 \]
\[ \varphi(t) = \alpha |t|/(1 + \alpha |t|), \quad \varphi'(0^+) = \alpha \]
TV energy: $\mathcal{F}_v(u) = \|Au - v\|^2 + \beta \mathrm{TV}(u)$

Questions to clarify the main property

Let \( u_o \in \mathbb{R} \) and \( \text{pdf}(u_o) = \frac{1}{2}e^{-|u_o|} \) (Laplacian distribution)

**Question 8** Give \( \text{Pr}(u_o = 0) \).

Let \( v = u_o + n \) where \( \text{pdf}(n) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{n^2}{2\sigma^2}} \) (centered Gaussian distribution)

The corresponding MAP energy to recover \( u_o \) from \( v \) reads as

\[
F_v(u) = \frac{1}{2}(u - v)^2 + \beta |u| \quad \text{for} \quad \beta = \frac{1}{\sigma^2}
\]

**Question 9** Give the minimizer function \( \mathcal{U} \) for \( F_v \).

Useful reminder on p. 24.

**Question 10** Determine the set \( \{ \nu \in \mathbb{R} : \mathcal{U}(\nu) = 0 \} \). Comment the result.
Figure 7. Rectified stereo image pair and the ground truth disparity. Light gray pixels indicate structures near to the camera, and black pixels correspond to unknown disparity values.

Image credits to the authors: Pock, Cremers, Bischof, and Chambolle “Global Solutions of Variational Models with Convex Regularization”, SIIMS 3(4) 2010, pp. 1122-1145
Minimization of $\mathcal{F}_v(u) = \|u - v\|_2^2 + \beta \text{TV}(u)$, $\beta = 100$ and $\beta = 180$
Questions relevant to the Potts model (see p. 12)

Here \( \varphi(t) = \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } t \neq 0 
\end{cases} \)

Question 11 Compute the global minimizer of \( \mathcal{F}_v(u) = (u - v)^2 + \beta \varphi(u) \) for \( u, v \in \mathbb{R} \) and \( \beta > 0 \), according to the value of \( v \).

Consider \( \mathcal{F}_v(u) = \|u - v\|_2^2 + \beta \sum_{i=1}^{p} \varphi(u[i]) \) for \( \beta > 0 \) and \( u, v \in \mathbb{R}^p \).

Note: \( \sum_{i=1}^{p} \varphi(u[i]) = \#\{i : u[i] \neq 0\} = \ell_0(u) \) is the counting norm.

The global minimizer function \( \mathcal{U} : \mathbb{R}^p \to \mathbb{R}^p \) for \( \mathcal{F}_v \) has \( p \) components which depend on \( v \).

Question 12 Compute each component \( \mathcal{U}_i \)

Question 13 Let \( h \subset \{1, \cdots, p\} \). Determine the subset \( \mathcal{O}_h \subset \mathbb{R}^p \) such that if \( v \in \mathcal{O}_h \) then the global minimizer \( \hat{u} \) of \( \mathcal{F}_v \) satisfies \( \hat{u}[i] = 0, \ \forall \ i \in h \) and \( \hat{u}[i] \neq 0 \) if \( i \notin h \).
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4 Minimizers relevant to non-smooth data-fidelity

General case [Nikolova 02]

\[ F_v(u) = \sum_i \psi(|a_i u - v[i]|) + \beta \Phi(u), \quad a_i = A_{\text{row } i}, \Phi \in \mathcal{C}^m, \psi \in \mathcal{C}^m(\mathbb{R}^*_+), \psi'(0^+) > 0 \]

Let \( \hat{u} \) be a (local) minimizer of \( F_v \). Set \( \hat{h} =: \{i : a_i \hat{u} = v[i]\} \).

Then \( \exists O \subset \mathbb{R}^q \) open, \( \exists U \in \mathcal{C}^{m-1} \) (local) minimizer function so that

\[ v' \in O, \quad \hat{u}' = U(v') \quad \Rightarrow \quad a_i \hat{u}' = v[i], \quad \forall i \in \hat{h} \]

\[ \hat{h} \subset \{1, .., q\} \quad \mathcal{O}_{\hat{h}} := \{v \in \mathbb{R}^q : a_i U(v) = v_i, \forall i \in \hat{h}\} \quad \Rightarrow \quad L^q(\mathcal{O}_{\hat{h}}) > 0 \]

(Local) minimizers \( \hat{u} \) of \( F_v \) achieve an exact fit to (noisy) data

\[ a_i \hat{u} = v[i] \quad \text{for a certain number of indexes } i \]

Property fails if \( F \) is fully smooth, except for \( v \in N \) where \( N \) is closed and \( L^q(N) = 0 \).
Question 14: Suggest cases when you would like that your minimizer obeys this property.

Question 15: Compute the minimizer of $F_v(u) = |u - v| + \beta u^2$ for $u, v \in \mathbb{R}$ and $\beta > 0$.

Question 16: Can you find a relationship between the properties of the minimizer when $\phi'(0^+) > 0$ (chapter 3, p. 26) and when $\psi'(0^+) > 0$ (chapter 4, p. 35).
$$\mathcal{F}_v(u) = \sum_i |u[i] - v[i]| + \beta \sum_{j \in N_i} |u[i] - u[j]|^{1.1}$$
Restoration \( \hat{u} \) for \( \beta = 0.25 \)

Residuals \( v - \hat{u} \)

\[
F_v(u) = \sum_i |u[i] - v[i]| + \beta \sum_{j \in N_i} |u[i] - u[j]|^{1.1}
\]

Restoration \( \hat{u} \) for \( \beta = 0.2 \)

Residuals \( v - \hat{u} \)

TV-like energy: \( F_v(u) = \sum_i (u[i] - v[i])^2 + \beta \sum_{j \in N_i} |u[i] - u[j]| \)
Detection and cleaning of outliers using $\ell_1$ data-fidelity

\[ F_v(u) = \sum_{i=1}^{p} |u[i] - v[i]| + \frac{\beta}{2} \sum_{i=1}^{p} \sum_{j \in N_i} \varphi(|u[i] - u[j]|) \]

$\varphi$: smooth, convex, edge-preserving

Assumptions:

\[
\begin{array}{l}
\text{data } v \text{ contain uncorrupted samples } v[i] \\
v[i] \text{ is outlier if } |v[i] - v[j]| \gg 0, \quad \forall j \in N_i
\end{array}
\]

$v \in \mathbb{R}^p \Rightarrow \hat{u} = \arg \min_u F_v(u) \quad \begin{cases} v[i] \text{ is regular if } i \in \hat{h} \\ v[i] \text{ is outlier if } i \in \hat{h}^c \end{cases}$

Outlier detector: $v \rightarrow \hat{h}^c(v) = \{ i : \hat{u}[i] \neq v[i] \}$

Smoothing: $\hat{u}[i]$ for $i \in \hat{h}^c$ = estimate of the outlier

Justification based on the properties of $\hat{u}$
L. Bar, A. Brook, N. Sochen and N. Kiryati,
“Deblurring of Color Images Corrupted by Impulsive Noise”,
IEEE Trans. on Image Processing, 2007

\[ \mathcal{F}_v(u) = \|Au - v\|_1 + \beta \Phi(u) \]

blurred, noisy (r.-v.)                      zoom - restored
Recovery of frame coefficients using $\ell_1$ data-fitting

- Data: $v = u_o + \text{noise}$
- Frame coefficients: $y = Wv = Wu_o + \text{noise}$
- Hard thresholding $y_T[i] := \begin{cases} 0 & \text{if } |y[i]| \leq T \\ y[i] & \text{if } |y[i]| > T \end{cases}$
  
  keeps relevant information if $T$ small
- $\tilde{u} = \tilde{W} y_T$ — Gibbs oscillations and wavelet-shaped artifacts
- Hybrid energy methods—combine fitting to $y_T$ with prior $\Phi(u)$

[Bobichon, Bijaoui 97], [Coifman, Sowa 00], [Durand, Froment 03]...
Desiderata: $\mathcal{F}_y$ convex and

Keep $\hat{x}[i] = y_T[i]$  

<table>
<thead>
<tr>
<th>Keep $\hat{x}[i] = y_T[i]$</th>
<th>Restore $\hat{x}[i] \neq y_T[i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>significant coefs: $y[i] \approx (W u_o)[i]$</td>
<td>outliers: $</td>
</tr>
<tr>
<td>thresholded coefs: $(W u_o)[i] \approx 0$</td>
<td>edge coefs: $</td>
</tr>
</tbody>
</table>

Then:

$$\text{minimize} \quad \mathcal{F}_y(x) = \sum_i \lambda_i |(x - y_T)[i]| + \int_{\Omega} \varphi(|\nabla \tilde{W} x|) \Rightarrow \hat{x}$$

$$\hat{u} = \tilde{W} \hat{x} \quad \text{for} \quad \tilde{W} \quad \text{left inverse, } \varphi \text{ edge-preserving}$$

**Question 17** Explain why the minimizers of $\mathcal{F}_y$ fulfill the desiderata.

**Hint:** "good" coefficients fitted exactly, "bad" coefficients corrected according to the prior.
Original and data

Sure-shrink method

Hard thresholding

Total variation

The proposed method

Magnitude of coefficients

Restored signal (—), original signal (−−).
Fast 2-stage restoration under impulse noise  

[R.Chan, Nikolova et al. 04,05,08]

1. Approximate the outlier-detection stage by rank-order filter
   (e.g. adaptive or center-weighted median)
   Corrupted pixels \( \hat{h}^c = \{i : \hat{v}[i] \neq v[i]\} \) where \( \hat{v} \) = Rank-Order Filter \( (v) \)
   \( \Rightarrow \) improve speed and accuracy

2. Restore \( \hat{u} \) (denoise, deblur) using an edge-preserving energy method
   subject to \( a_i \hat{u} = v[i] \) for all \( i \in \hat{h} \)
One-step real-time dejittering of digital video

- Image \( u \in \mathbb{R}^{m \times n} \), rows \( u_i \), its pixels \( u_i[j] \)

- Data \( v_i[j] = u_i[j + d_i] \), \( d_i \) integer, \( |d_i| \leq M \), typically \( M \leq 20 \).

- Restore \( \hat{u} \equiv \text{restore} \, \hat{d}_i \), \( 1 \leq i \leq m \)

The gray-values of the columns of natural images can be seen as large pieces of \( 2^{\text{nd}} \) (or \( 3^{\text{rd}} \)) order polynomials which is false for their jittered versions.
Each column \( \hat{u}_i \) is restored using \( \hat{d}_i = \arg \min_{|d_i| \leq N} \mathcal{F}(d_i) \)

\[
\mathcal{F}(d_i) = \sum_{j=N+1}^{c-N} |v_i[j + d_i] - 2\hat{u}_{i-1}[j] + \hat{u}_{i-2}[j]|^\alpha, \quad \alpha \in \{0.5, 1\}, \quad N > M
\]

**Question 18** Explain why the minimizers of \( \mathcal{F} \) can solve the problem as stated.

**Question 19** What changes if \( \alpha = 1 \) or if \( \alpha = 0.5 \)?

**Question 20** Is it easy to solve the numerical problem?

A Monte-Carlo experiment shows that in almost all cases, \( \alpha = 0.5 \) is better.

Jittered, \([-20, 20]\) \hspace{1cm} \( \alpha = 1 \) \hspace{1cm} Jitter: \( 6 \sin \left( \frac{n}{4} \right) \) \hspace{1cm} \( \alpha = 1 \equiv \text{Original} \)
Jittered \{-8, \ldots, 8\} \qquad \text{Original image} \qquad \alpha = 1 \qquad \text{Zooms}

(512 \times 512) \quad \text{Jitter} \, M = 6 \; \alpha \in \{1, \frac{1}{2}\} = \text{Original Lena (256 \times 256)} \qquad \text{Jitter} \{-6, \ldots, 6\} \quad \alpha \in \{1, \frac{1}{2}\}
Jitter \{-15, \ldots, 15\}

\[ \alpha = 1, \alpha = 0.5 \]

Original image
Jitter Jittered Image Bayesian TV Bake & Shake

Original Our: $\alpha=0.5$ Our: Error $u_o - \hat{u}$

[Kokaram98, Laborelli03, Shen04, Kang06, Scherzer11]
5. Comparison with Fully Smooth Energies

\[ \mathcal{F}_v(u) = \Psi(u, v) + \beta \Phi(u), \quad \mathcal{F} \in C^m > 2 + \text{easy assumptions}. \]

If \( h \neq \emptyset \Rightarrow \)

\[
\{ v \in \mathbb{R}^q : \mathcal{F}_v - \text{minimum at } \hat{u}, \ G_i \hat{u} = 0, \ \forall i \in h \} \quad \text{closed and}
\]

\[
\{ v \in \mathbb{R}^q : \mathcal{F}_v - \text{minimum at } \hat{u}, \ a_i \hat{u} = v_i, \ \forall i \in h \} \quad \text{negligible in } \mathbb{R}^q
\]

For \( \mathcal{F}_v \) smooth, the chance that noisy data \( v \) yield a minimizer \( \hat{u} \) of \( \mathcal{F}_v \) which for some \( i \) satisfies exactly \( G_i \hat{u} = 0 \) or \( a_i \hat{u} = v_i \) is negligible

Nearly all \( v \in \mathbb{R}^q \) lead to \( \hat{u} = \mathcal{U}(v) \) satisfying \( G_i \hat{u} \neq 0, \ \forall i \) and \( a_i \hat{u} \neq v_i, \ \forall i \)

**Question 21** What are the consequences if one approximates a nonsmooth energy by a smooth energy?
Questions to clarify the theoretical results

Let $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$.

Consider that $A \in \mathbb{R}^{q \times p}$ and $G \in \mathbb{R}^{r \times p}$ satisfy $\ker(A) \cap \ker(G) = \{0\}$.

$$F_v(u) = \|Au - v\|_2^2 + \beta \|Gu\|_2^2 \quad \text{for} \quad \beta > 0$$

**Question 22** Calculate $\nabla F_v(u)$.

**Question 23** Determine the minimizer function $U$.

Let $G_i \in \mathbb{R}^{1 \times p}$ denote the $i$th row of $G$.

**Question 24** Characterize the set $K = \{\nu \in \mathbb{R}^p : G_i U(\nu) = 0\}$.

Let $a_i \in \mathbb{R}^{1 \times p}$ denote the $i$th row of $A$.

**Question 25** Characterize the set $L = \{\nu \in \mathbb{R}^p : a_i U(\nu) = \nu[i]\}$.
Summer School 2014: Inverse Problem and Image Processing

Tutorial: Inverse modeling in inverse problems using optimization

Outline

1. Energy minimization methods (p. 7)
2. Regularity results (p. 17)
3. Non-smooth regularization – minimizers are sparse in a given subspace (p. 26)
4. Non-smooth data-fidelity – minimizers fit exactly some data entries (p. 35)
5. Comparison with Fully Smooth Energies (p. 51)
6. Non-convex regularization – edges are sharp
7. Nonsmooth data-fidelity and regularization – peculiar features (p. 62)
8. Fully smoothed $\ell_1$–TV models – bounding the residual (p. 83)
9. Inverse modeling and Bayesian MAP – there is distortion (p. 98)
10. Some References (p. 103)
6 Nonconvex Regularization: Why Edges are Sharp? [Nikolova 04, 10]

\[ \mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum_{i \in J} \varphi(\|G_i u\|) \]

\[ J = \{1, \cdots, r\} \]

**Standard assumptions on \( \varphi \):** 
- \( C^2 \) on \( \mathbb{R}_+ \) and \( \lim_{t \to \infty} \varphi''(t) = 0 \), as well as:
  - \( \varphi'(0) = 0 \) (\( \Phi \) is smooth)
  - \( \varphi'(0^+) > 0 \) (\( \Phi \) is nonsmooth)
Illustration on $\mathbb{R}$

$\mathcal{F}_v(u) = (u - v)^2 + \beta \varphi(|u|), \ u, v \in \mathbb{R}$

No local minimizer in $(\theta_0, \theta_1)$

$\exists \xi_0 > 0, \ \exists \xi_1 > \xi_0$

$|v| \leq \xi_1 \Rightarrow |\hat{u}_0| \leq \theta_0$

strong smoothing

$|v| \geq \xi_0 \Rightarrow |\hat{u}_1| \geq \theta_1$

loose smoothing

$\exists \xi \in (\xi_0, \xi_1)$

$|v| \leq \xi \Rightarrow \text{global minimizer} = \hat{u}_0$ (strong smoothing)

$|v| \geq \xi \Rightarrow \text{global minimizer} = \hat{u}_1$ (loose smoothing)

For $v = \xi$ the global minimizer jumps from $\hat{u}_0$ to $\hat{u}_1 \equiv \text{decision for an “edge”}$

Since [Geman2 1984] various nonconvex $\Phi$ to produce minimizers with smooth regions and sharp edges
Sharp edge property

There exist $\theta_0 \geq 0$ and $\theta_1 > \theta_0$ such that any (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ satisfies

$$\text{either } \|G_i\hat{u}\| \leq \theta_0 \text{ or } \|G_i\hat{u}\| \geq \theta_1 \quad \forall i \in J$$

$$\hat{h}_0 = \{ i : \|G_i\hat{u}\| \leq \theta_0 \} \quad \text{homogeneous regions}$$
$$\hat{h}_1 = \{ i : \|G_i\hat{u}\| \geq \theta_1 \} \quad \text{edges}$$

When $\beta$ increases, then $\theta_0$ decreases and $\theta_1$ increases.

In particular

$$\varphi'(0^+) > 0 \Rightarrow \theta_0 = 0 \quad \text{fully segmented image} \quad (G_i\hat{u} = 0, \forall i \in \hat{h}_0)$$

**Question 26** Explain the prior model involved in $\mathcal{F}_v$ when $\varphi$ is nonconvex with $\varphi'(0) = 0$ and with $\varphi'(0^+) > 0$. 
Image Reconstruction in Emission Tomography

Original phantom

Emission tomography simulated data

φ is smooth (Huber function)

φ(t) = t/(α + t) (non-smooth, non-convex)

Reconstructions using \( \mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{j \in \mathcal{N}_i} \phi(|u[i] - u[j]|), \) \( \Psi = \text{smooth, convex} \)
Selection for the global minimizer

**Additional assumptions:** \( \| \varphi \|_{\infty} < \infty, \{ G_i \} — 1^{st}\)-order differences, \( A^*A \) invertible

\[
\mathbb{1}_{\Sigma i} = \begin{cases} 
1 & \text{if } i \in \Sigma \subset \{1, .., p\} \\
0 & \text{else} 
\end{cases}
\]

Original: \( u_o = \xi \mathbb{1}_{\Sigma} \), \( \xi > 0 \)

Data: \( v = \xi A \mathbb{1}_{\Sigma} = Au_o \)

\( \hat{u} = \text{global minimizer of } \mathcal{F}_v \)

**Sketch of the results**

\( \exists \xi_1 > 0 \) such that \( \xi > \xi_1 \Rightarrow \hat{u} — \text{perfect edges} \)

Moreover:

- \( \Phi \) non smooth, then \( \xi > \xi_1 \Rightarrow \hat{u} = c u_o, \ c < 1, \ \lim_{\xi \to \infty} c = 1 \)
- \( \varphi(t) = \eta, \ t \geq \eta, \) then \( \xi > \xi_1 \Rightarrow \hat{u} = u_o \)

This holds true also for \( \varphi(t) = \min\{\alpha t^2, 1\} \) and for \( \varphi(t) = \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } t \neq 0 
\end{cases} \)
Comparison with Convex Edge-Preserving Regularization

Data \( v = u_o + n \)

\( \varphi(t) = |t| \)

\( \varphi(t) = \alpha|t|/(1 + \alpha|t|) \)

original data \( \varphi(t) = |t|^{1.4} \)

\( \varphi(t) = \min\{\alpha t^2, 1\} \)

Question 27  Why edges are sharper when \( \varphi \) is nonconvex?
$F_v(u) = (u - v)^2 + \beta \frac{\alpha |u|}{(1 + \alpha |u|)}$

global function ($\bullet$)

$F_v(u) = (u - v)^2 + \beta \frac{\alpha u^2}{(1 + \alpha u^2)}$

global minimizer functions ($\bullet$)

$F_v(u) = (u - v)^2 + \beta \sqrt{\alpha + u^2}$

unique minimizer function ($\bullet$)

Each blue curve curve: $u \rightarrow F_v(u)$ for $v \in \{0, 2, \cdots\}$

**Question 28** How to describe the global minimizer when $v$ increases?
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Consequence of §3 and §4: if $\Phi$ and $\Psi$ non-smooth, 
\[
\begin{cases}
G_i \hat{u} = 0 & \text{for } i \in \hat{h}_\Phi \neq \emptyset \\
a_i \hat{u} = v[i] & \text{for } i \in \hat{h}_\Psi \neq \emptyset
\end{cases}
\]

The $L_1$-TV energy


\[
\mathcal{F}_v(u) = \|u - 1_\Omega\|_1 + \beta \int_{\mathbb{R}^d} \|\nabla u(x)\|_2 \, dx \quad \text{where} \quad 1_\Omega(x) := \begin{cases} 
1 & \text{if } x \in \Omega \\
0 & \text{else}
\end{cases}
\]

- $\exists \hat{u} = 1_\Sigma$ ($\Omega$ convex $\Rightarrow$ $\Sigma \subset \Omega$ and $\hat{u}$ unique for almost every $\beta > 0$)
- contrast invariance: if $\hat{u}$ minimizes for $v = 1_\Omega$ then $c\hat{u}$ minimizes $\mathcal{F}_{cv}$
  the contrast of image features is more important than their shapes
- critical values $\beta^*$
  \[
  \begin{cases} 
  \beta < \beta^* & \Rightarrow \text{objects in } \hat{u} \text{ with good contrast} \\
  \beta > \beta^* & \Rightarrow \text{they suddenly disappear}
  \end{cases}
  \]
  $\Rightarrow$ data-driven scale selection
Binary images by $\text{L1} - \text{TV}$

Classical approach to find a binary image $\hat{u} = 1_{\hat{\Sigma}}$ from binary data $1_{\Omega}$, $\Omega \subset \mathbb{R}^2$

$$\hat{\Sigma} = \arg \min_{\Sigma} \left\{ \| 1_{\Sigma} - 1_{\Omega} \|_2^2 + \beta \text{TV}(1_{\Sigma}) \right\}$$

nonconvex problem \( (\star) \)

usual techniques (curve evolution, level-sets) fail

$$\hat{\Sigma} \text{ solves } (\star) \iff \hat{u} = 1_{\hat{\Sigma}} \text{ minimizes } \| u - 1_{\Omega} \|_1 + \beta \text{ TV}(u) \text{ (convex)}$$

Data

Restored
Multiplicative noise removal on Frame coefficients [Durand, Fadili, Nikolova 09]

Multiplicative noise arises in various active imaging systems e.g. synthetic aperture radar

- Original image: \( S_o \)
- One shot: \( \Sigma_k = S_o \eta_k \)
- Data: \( \Sigma = \frac{1}{K} \sum_{k=1}^{K} \Sigma_k = S_o \frac{1}{K} \sum_{k=1}^{K} \eta_k = S_o \eta \) where \( \text{pdf}(\eta) = \text{Gamma density} \)
- Log-data: \( v = \log \Sigma = \log S_o + \log \eta = u_0 + n \)
- Frame Coefficients: \( y = Wv = Wu_0 + Wn \) (W curvelets)

Question 29  
Comment the noise distribution of \( Wn \)
• Hard Thresholding:  \( y_T[i] = \begin{cases} 0 & \text{if } |y[i]| \leq T, \\ y[i] & \text{otherwise} \end{cases} \) \( \forall i \in I, \ T > 0 \) (suboptimal).

\[
I_1 = \{i \in I : |y[i]| > T\} \text{ and } I_0 = I \setminus I_1
\]

• Restored coefficients:  \( \hat{x} = \arg \min_x \mathcal{F}_y(x) \) (\( \ell_1 - \text{TV energy} \))

\[
\mathcal{F}_y(x) = \lambda_0 \sum_{i \in I_0} |x[i]| + \lambda_1 \sum_{i \in I_1} |x[i] - y[i]| + \|\widetilde{W}x\|_{\text{TV}}
\]

\[
\hat{S} = B \exp(\widetilde{W}\hat{x}), \text{ where } \widetilde{W} \text{ left inverse, } B \text{ bias correction}
\]

Question 30  Explain the job the minimizer \( \hat{x} \) of \( \mathcal{F}_y \) should do.

Some comparisons

• BS [Chesneau,Fadili,Starck 08]: Block-Stein thresholds the curvelet coefficients, \( \approx \) minimax(large class of images with additive noises), optimal threshold \( \mathcal{X} = 4.50524 \)

• AA [Aubert,Aujol 08]: \( \Psi = -\text{Log-Likelihood}(\Sigma), \Phi = \text{TV}(\Sigma) \) (i.e. \( \mathcal{F}_v \equiv \text{MAP for } \Sigma \))

• SO [Shi,Osher 08]: relaxed inverse scale-space for \( \mathcal{F}_v(u) = \|v - u\|_2^2 + \beta \text{TV}(u) \approx \text{MAP}(u) \)

Stopping rule: \( k^* = \max\{k \in \mathbb{N} : \text{Var}(u^{(k)} - u_o) \geq \text{Var}(n)\} \).

Monte-Carlo comparative experiment confirms the proposed method
Noisy Fields $K = 1$ (512×512)

SO: PSNR=9.59, MAE=196

AA: PSNR=15.74, MAE=76.66

BS: PSNR=22.52, MAE=35.22

Fields (original)

Our: PSNR=22.89, MAE=33.67
Noisy $K = 10$

BS: PSNR=27.24, MAE=19.61

Fields (original)

Our: PSNR=28.04, MAE=18.19
Noisy City $K = 1 \ (512 \times 512)$

SO: PSNR=18.39, MAE=24.08

AA: PSNR=22.18, MAE=13.71

BS: PSNR=22.25, MAE=13.96

City (original)

Our: PSNR=22.64, MAE=13.39
Noisy $K = 4$

SO: PSNR=24.40, MAE=10.76

AA: PSNR=24.55, MAE=10.06

BS: PSNR=24.92, MAE=9.87

City (original)

Our: PSNR=25.84, MAE=9.09
C. Clason, B. Jin, K. Kunisch
“Duality-based splitting for fast $\ell_1 - \text{TV}$ image restoration”, 2012,
http://math.uni-graz.at/optcon/projects/clason3/

Scanning transmission electron microscopy ($2048 \times 2048$ image)

true image  noisy image  restoration
**Motivation**

- This family of objective functions has never been considered before
- $\mathcal{F}_v$ can be seen as an extension of $L1 - TV$
- $\hat{u}$—(local) minimizer of $\mathcal{F}_v$ \(\Rightarrow\) many $i, j$ such that \(a_i\hat{u} = v[i]\) and \(G_j\hat{u} = 0\)
Minimizers of \[ \mathcal{F}_v(u) = \|u - v\|_1 + \beta \sum_{i=1}^{p-1} \varphi(|u[i+1] - u[i]|) \]

\[ \varphi(t) = \frac{\alpha t}{\alpha t + 1} \text{ for } \alpha = 4 \]

\[ \varphi(t) = \ln(\alpha t + 1) \text{ for } \alpha = 2 \]

\[ \beta \in \{78, \ldots, 156\} \]

\[ \beta \in 0.1 \times \{10, \ldots, 14\} \]

\[ \beta \in \{157, \ldots, 400\} \]

\[ \beta \in 0.1 \times \{16, \ldots, 30\} \]

Data samples (ooo), Minimizer samples \( \hat{u}[i] \) (+++).
(a) \( \varphi(t) = \frac{\alpha t}{\alpha t + 1}, \alpha = 4, \beta = 3 \)

(b) \( \varphi(t) = 1 - \alpha^t, \alpha = 0.1, \beta = 2.5 \)

(c) \( \varphi(t) = \ln(\alpha t + 1), \alpha = 2, \beta = 1.3 \)

(d) \( \varphi(t) = (t + 0.1)^\alpha, \alpha = 0.5, \beta = 1.4 \)

**Denoising:** Data samples (○○○) are corrupted with Gaussian noise. Minimizer samples \( \hat{u}[i] \) (++++). Original (−−−). \( \beta \) — the largest value so that the gate at 71 survives.
Constant pieces—solid black line.

Data points $v[i]$ fitted exactly by the minimizer $\hat{u}$ (♦).
\[ \varphi(t) = t, \beta = 0.8 \quad (\ell_1 - TV) \]

the convex relaxation of \( F_v \)

the minimizer for \( \varphi(t) = \frac{\alpha t}{\alpha t + 1}, \alpha = 4, \beta = 3 \)

closest to \((\ell_1 - TV)\)

error for \( \varphi(t) = \frac{\alpha t}{\alpha t + 1}, \alpha = 4, \beta = 3 \)

\[ \|\text{original} - \hat{u}\|_\infty = 0.24 \]

\[ \varphi(t) = \frac{\alpha t}{\alpha t + 1}, \alpha = 4, \beta = 3 \]

original \(\in [0, 12]\), data \(v \in [-0.6, 12.9]\)
On the figures, $\hat{u}$ are global minimizers of $F_v$ (Viterbi algorithm)

**Question 31** Can you sketch the main properties of the minimizers of $F_v$?

**Question 32** What seems being the role of the asymptotic of $\varphi$?

**Numerical evidence:**

Critical values $\beta_1, \ldots, \beta_n$ such that

- $\beta \in [\beta_i, \beta_{i+1}) \Rightarrow$ the minimizer remains unchanged
- $\beta \geq \beta_{i+1} \Rightarrow$ the minimizer is simplified

Result proven (under conditions) for the minimizers of $L_1 - TV$ in [Chan, Esedoglu 2005]
Given $v \in \mathbb{R}$ consider the function

$$
\mathcal{F}_v(u) = |u - v| + \beta \varphi(|u|) \quad \text{for} \quad \varphi(u) = \frac{\alpha u}{1 + \alpha u} \quad u \in \mathbb{R}, \quad \beta > 0
$$

Question 33 Does $\mathcal{F}_v$ have a global minimizer for any $v$? Explain.

Question 34 Determine $\varphi''(u)$ for $u \in \mathbb{R} \setminus \{0\}$.

Question 35 Show that $\forall v \in \mathbb{R}$, any minimizer $\hat{u}$ of $\mathcal{F}_v$ obeys $\hat{u} \in \{0, v\}$.

Question 36 Can you extend this result to the other $\varphi$ on p. 71?
• $\mathcal{F}_v$ does have global minimizers, for any $\{a_i\}$, for any $v$ and for any $\beta > 0$.

• Let $\hat{u}$ be a (local) minimizer of $\mathcal{F}_v$. Set

$$
\hat{I}_0 = \{ i \in I : a_i \hat{u} = v[i] \}
$$

$$
\hat{J}_0 = \{ j \in J : G_j \hat{u} = 0 \}
$$

$\hat{u}$ is the unique point solving the linear system

$$
\begin{cases}
a_i \hat{u} = v[i] & \forall i \in \hat{I}_0 \\
G_j \hat{u} = 0 & \forall j \in \hat{J}_0
\end{cases}
$$

Each pixel of a (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ is involved in (at least) one equation $a_i \hat{u} = v[i]$, or in (at least) one equation $G_j \hat{u} = 0$, or in both types of equations.

• “Contrast invariance” of (local) minimizers

• The matrix with rows $(a_i, \forall i \in \hat{I}_0, \ G_j, \forall j \in \hat{J}_0)$ has full column rank

• All (local) minimizers of $\mathcal{F}_v$ are strict
MR Image Reconstruction from Highly Undersampled Data

Reconstructed images from 7% **noisy** randomly selected samples in the \( k \)-space.

Our method for \( \varphi(t) = \frac{\alpha t}{\alpha t + 1} \).
MR Image Reconstruction from Highly Undersampled Data

Reconstructed images from 5% noisy randomly selected samples in the $k$-space.

Our method for $\varphi(t) = \frac{\alpha t}{\alpha t + 1}$. 
Cartoon

Observed

\[ \ell_1\text{-TV} \]

Our method, \( \varphi(t) = \frac{\alpha t}{\alpha t + 1} \)
Outline

1. Energy minimization methods (p. 7)
2. Regularity results (p. 17)
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4. Non-smooth data-fidelity – minimizers fit exactly some data entries (p. 35)
5. Comparison with Fully Smooth Energies (p. 51)
6. Non-convex regularization – edges are sharp (p. 54)
7. Nonsmooth data-fidelity and regularization – peculiar features (p. 62)
8. Fully smoothed $\ell_1$–TV models – bounding the residual
9. Inverse modeling and Bayesian MAP – there is distortion (p. 98)
10. Some References (p. 103)
8. Fully smoothed $\ell_1 - \text{TV}$

\[
\mathcal{F}_v(u) = \Psi(u, v) + \beta \Phi(u), \quad \beta > 0
\]

\[
\Psi(u, v) = \sum_{i=1}^{p} \psi(u[i] - v[i]) \quad \text{and} \quad \Phi(u) = \sum_i \varphi(|G_i u|)
\]

\[
\psi(\cdot) := \psi(\cdot, \alpha_1)
\]

\[
\varphi(\cdot) := \varphi(\cdot, \alpha_2)
\]

\[
(\alpha_1, \alpha_2) > 0
\]

\(\{G_i \in \mathbb{R}^{1 \times p}\} - \text{forward discretization:} \)

\(\mathcal{N}_4\) Only vertical and horizontal differences;

\(\mathcal{N}_8\) Diagonal differences are added.

\((\psi, \varphi)\) belong to the family of functions \(\theta(\cdot, \alpha) : \mathbb{R} \to \mathbb{R}\) satisfying

**H 1** For any \(\alpha > 0\) fixed, \(\theta(\cdot, \alpha)\) is \(C^{s \geq 2}\)-continuous, even and \(\theta''(t, \alpha) > 0, \forall t \in \mathbb{R}\).

**H 2** For any \(\alpha > 0\) fixed, \(|\theta'(t, \alpha)| < 1\) and for \(t > 0\) fixed, it is strictly decreasing in \(\alpha > 0\)

\[
\alpha > 0 \quad \Rightarrow \quad \lim_{t \to \infty} \theta'(t, \alpha) = 1
\]

\[
t \in \mathbb{R} \quad \Rightarrow \quad \lim_{\alpha \to 0} \theta'(t, \alpha) = 1 \quad \text{and} \quad \lim_{\alpha \to \infty} \theta'(t, \alpha) = 0.
\]

\(\Rightarrow \mathcal{F}_v\) is a fully smoothed $\ell_1 - \text{TV}$ energy.
<table>
<thead>
<tr>
<th></th>
<th>( \theta )</th>
<th>( \theta' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>f1</td>
<td>( \sqrt{t^2 + \alpha} )</td>
<td>( \frac{t}{\sqrt{t^2 + \alpha}} )</td>
</tr>
<tr>
<td>f2</td>
<td>( \alpha \log \left( \cosh \left( \frac{t}{\alpha} \right) \right) )</td>
<td>( \tanh \left( \frac{t}{\alpha} \right) )</td>
</tr>
<tr>
<td>f3</td>
<td>(</td>
<td>t</td>
</tr>
</tbody>
</table>

Choices for \( \theta(\cdot, \alpha) \) obeying H1 and H2. When \( \alpha \searrow 0 \), \( \theta(\cdot, \alpha) \) becomes stiff near the origin.

\[
\theta(t) = \sqrt{t^2 + \alpha} \quad \theta'(t) = \frac{t}{\sqrt{t^2 + \alpha}} \quad (\theta')^{-1}(y) = y \sqrt{\frac{\alpha}{1-y^2}}
\]

Plots of f1 for \( \alpha = 0.05 \) (---) and for \( \alpha = 0.5 \) (---).
The minimizers \( \hat{u} \) of \( F_v \) can decrease the quantization noise.
• For any $\beta > 0$, $\mathcal{F}_v(\mathbb{R}^p)$ has a unique minimizer function $\mathcal{U} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ which is $\mathcal{C}^{s-1}$.

Define $\mathcal{G} := \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \left\{ g \in \mathbb{R}^{1 \times p} : g[i] = -g[j] = 1, \ i \neq j, \ g[k] = 0 \text{ if } k \not\in \{i, j\} \right\}$

All difference operators $G_i$ belong to $\mathcal{G}$.

$$N_G := \bigcup_{g \in \mathcal{G}} \left\{ v \in \mathbb{R}^p : g \mathcal{U}(v) = 0 \right\} \quad \text{and} \quad N_I := \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \left\{ v \in \mathbb{R}^p : \mathcal{U}_i(v) = v[j] \right\}$$

**Question 37** How to interpret the sets $N_G$ and $N_I$?

• The sets $N_G$ and $N_I$ are closed in $\mathbb{R}^p$ and obey

$$\mathbb{L}^p(N_G) = 0 \quad \text{and} \quad \mathbb{L}^p(N_I) = 0$$

The property is true for any $\beta > 0$ and $(\alpha_1, \alpha_2) > 0$. 
\( \mathbb{R}^p \setminus (N_G \cup N_I) \) is open and dense in \( \mathbb{R}^p \).

The elements of \((N_G \cup N_I)\) are highly exceptional in \( \mathbb{R}^p \).

The minimizers \( \hat{u} \) of \( \mathcal{F}_v \) generically satisfy \( \hat{u}[i] \neq \hat{u}[j] \) for any \((i, j)\) such that \( i \neq j \) and \( \hat{u}[i] \neq v[j] \) for any \((i, j)\).

The minimizers \( \hat{u} \) of \( \mathcal{F}_v \) have pixel values that are different from each other and different from any data pixel.

**Question 38** Describe the consequences if \( \ell_1 - \text{TV} \) is approximated by a smooth function like \( \mathcal{F}_v \).

Recall the illustration on p. 24 and the results in section 3 (p. 26) and section 4 (p. 35).
Further...

- For any $\alpha_1 > 0$ fixed, there is an inverse function $(\psi')^{-1} (\cdot, \alpha_1) : (-1, 1) \rightarrow \mathbb{R}$ which is odd, $C^{s-1}$ and strictly increasing.
  
  \[ \alpha_1 \mapsto (\psi')^{-1} (y, \alpha_1) \] is also strictly increasing on $(0, +\infty)$, for any $y \in (0, 1)$.

- Set $\eta := \|G\|_1$. Then

  \[ \beta \eta < 1 \quad \Rightarrow \quad \|\hat{u} - v\|_\infty \leq (\psi')^{-1} (\beta \eta, \alpha_1) \quad \forall \, v \in \mathbb{R}^p \]

- Also,

  \[ \|\hat{u} - v\|_\infty \nearrow (\psi')^{-1} (\beta \eta, \alpha_1) \text{ as } \alpha_2 \searrow 0. \]

**Full control on the bound $\|\hat{u} - v\|_\infty$.**

**Question 39** Can you suggest applications where the properties of $F_v$ are important?
Exact histogram specification

- \( v \) – input digital gray value \( m \times n \) image / stored as an \( p := mn \) vector

- \( v[i] \in \{0, \cdots, L - 1\} \quad \forall \ i \in \{1, \cdots, p\} \)
  
- 8-bit image \( \Rightarrow L = 256 \)

- Histogram of \( v \): \( H_v[k] = \frac{1}{p} \# \{ v[i] = k : i \in \{1, \cdots, p\} \} \quad \forall \ k \in \{0, \cdots, L - 1\} \)

- Target histogram: \( \zeta = (\zeta[1], \cdots, \zeta[L]) \)

- Goal of histogram specification (HS): convert \( v \) into \( \hat{u} \) so that \( H_{\hat{u}} = \zeta \)

  order the pixels in \( v \): \( i \prec j \) if \( v[i] < v[j] \)

  \[
  i_1 \prec i_2 \prec \cdots \prec i_{\zeta[1]} \prec \cdots \prec i_{p-\zeta[L]+1} \prec \cdots \prec i_p
  \]

  \( \zeta[1] \prec \cdots \prec \zeta[L-1] \)

- Ill-posed problem for digital (quantized) images since \( p \gg L \)

- An issue: obtain a meaningful total strict ordering of all pixels in \( v \)

Histogram equalization is a particular case of HS where \( \zeta[k] = p/L \quad \forall \ k \in \{0, \cdots L - 1\} \)
Histogram Equalization (HE) using Matlab and our ordering

input image

HE by "histeq"

HE by "sort"

HE our ordering

histograms

zooms
Modern sorting algorithms

For any pixel $v[i]$, extract $K$ auxiliary information, $a_k[i], k \in \{1, \cdots, K\}$, from $v$. Set $a_0 := v$. Then

$$i \prec j \text{ if } v[i] \leq v[j] \text{ and } a_k[i] < a_k[j] \text{ for some } k \in \{0, \cdots, K\}.$$ 

Local Mean Algorithm (LM) [Coltuc, Bolon, Chassery 06]

– If two pixels are equal and their local mean is the same, take a larger neighborhood.
– The procedure smooths edges and sorting often fails.

Wavelet Approach (WA) [Wan, Shi 07]

– Use wavelet coefficients from different subbands to order the pixels.
– Heavy and high level of failure.

Specialized variational approach (SVA) [Nikolova, Wen and R. Chan 12]

– Minimize $F_v$ for a parameter choice yielding $\|\hat{u} - v\|_\infty \lesssim 0.1$.
– Almost no failure, faithful order and fast algorithm. [Nikolova 13]
Some results using $F_v$ for color image enhancement

New fast histogram based color enhancement algorithm.  


club (1800 × 3200)

Hist.-based [NS 14]

Hist.-based [NS 14]

Perceptual [APBC 09]
boy-on-stones (800 × 800)

Hist.-based [NS 14]

Hist.-based [HYL 11]

Perceptual [BCPR 07]

Perceptual [APBC 09]

ACE [G 12]
Input “orchid” with a bad flashlight effect.
Goal – enhance the snake.
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6. Non-convex regularization – edges are sharp (p. 54)
7. Nonsmooth data-fidelity and regularization – peculiar features (p. 62)
8. Fully smoothed $\ell_1$–TV models – bounding the residual (p. 83)
9. Inverse modeling and Bayesian MAP – there is distortion
10. Some References (p. 103)
Inverse modeling and Bayesian MAP

MAP estimators to combine noisy data and prior

Bayesian approach: \( U, V \) random variables, events \( U = u, V = v \).

Likelihood \( f_{V|U}(v|u) \), Prior \( f_U(u) \propto \exp\{-\lambda \Phi(u)\} \), Posterior \( f_{U|V}(u|v) = f_{V|U}(v|u)f_U(u) \frac{1}{Z} \)

MAP \( \hat{u} = \) the most likely solution given the recorded data \( V = v \):
\[
\hat{u} = \arg \max_u f_{U|V}(u|v) = \arg \min_u \left( -\ln f_{V|U}(v|u) - \ln f_U(u) \right)
\]
\[
= \arg \min_u \left( \Psi(u, v) + \beta \Phi(u) \right)
\]

MAP is the most frequent way to combine models on data-acquisition and priors

Realist models for data-acquisition \( f_{V|U} \) and prior \( f_U \)
\[
\Rightarrow \hat{u} \text{ must be coherent with } f_{V|U} \text{ and } f_U
\]

In practice one needs that:
\[
\begin{align*}
U & \sim f_U \\
AU - V & \sim f_N
\end{align*} \Rightarrow
\begin{align*}
f_{\hat{U}} & \approx f_U \\
f_{\hat{N}} & \approx f_N, \quad \hat{N} \approx A\hat{U} - V
\end{align*}
\]

Our analytical results show that both models (\( f_{V|U} \) and \( f_U \)) are violated in a MAP estimate
**Example: MAP shrinkage**  

[Simoncelli99, Belge-Kilmer00, Antoniadis02]

- Noisy wavelet coefficients \( y = Wv = Wu_o + n = x_o + n, \quad n \sim \mathcal{N}(0, \sigma^2 I) \)

- Prior: \( x_o[i] \) are i.i.d., \( f(x_o[i]) = \frac{1}{Z} e^{-\lambda|x_o[i]|^\alpha} \) (Generalized Gaussian, GG)

Experiments have shown that \( \alpha \in (0, 1) \) for many real-world images

- MAP restoration \( \iff \hat{x}[i] = \arg \min_{t \in \mathbb{R}} ( (t - y[i])^2 + \lambda |t|^\alpha ), \quad \forall i \)

\((\alpha, \lambda, \sigma)\) fixed—10,000 independent trials:

1. sample \( x \sim f_X \) and \( n \sim \mathcal{N}(0, \sigma^2) \),
2. form \( y = x + n \),
3. compute the true MAP \( \hat{x} \)

---

**Graphs:**

- GG, \( \alpha = 1.2, \lambda = \frac{1}{2} \)
- The true MAP \( \hat{x} \)
- GG, \( \alpha = \frac{1}{2}, \lambda = 2 \)
- True MAP \( \hat{x} \)

- Noise \( \mathcal{N}(0, \sigma^2) \)
- Noise estimate
- Noise \( \mathcal{N}(0, \sigma^2) \)
- Noise estimate
Theoretical explanations

\[ V = AU + N \] and \( f_U|_V \) continuous \( \Rightarrow \)  
\[ \begin{align*} 
\Pr(G_i u = 0) & = 0, \quad \forall i \\
\Pr(a_i u = v_i) & = 0, \quad \forall i \\
\Pr(\theta_0 < \|G_i u\| < \theta_1) & > 0, \quad \forall i 
\end{align*} \]

The analytical results on \( \hat{u} = \arg\min_u F_v(u) = \text{MAP} \) yield:

- \( f_U \) continuous and non-smooth at 0, \( \varphi'(0^+) > 0 \)  
  \[ v \in O_{\hat{h}} \Rightarrow [G_i \hat{u} = 0, \forall i \in \hat{h}] \Rightarrow \Pr(G_i \hat{u} = 0, \forall i \in \hat{h}) > \Pr(v \in O_{\hat{h}}) > 0 \]
  The effective prior: \( G_i \hat{u} = 0 \) for many \( i \) (e.g. locally constant images)

- \( f_N \) continuous and nonsmooth at 0, \( \psi'(0^+) > 0 \)  
  \[ v \in O_{\hat{h}} \Rightarrow [a_i \hat{u} = v_i, \forall i \in \hat{h}] \Rightarrow \Pr(a_i \hat{u} = v_i, \forall i \in \hat{h}) > \Pr(V \in O_{\hat{h}}) > 0 \]
  The effective model: there are uncorrupted data entries.

- \( -\ln f_U \) (resp., \( \varphi \)) continuous and nonconvex \( \Rightarrow \Pr(\theta_0 < \|G_i \hat{U}\| < \theta_1) = 0, \forall i \)
  The effective prior: edges.

- \( -\ln f_U \) nonconvex, nonsmooth at 0, \( \varphi'(0^+) > 0 \) and \( \varphi'' \leq 0 \)  
  \[ \Pr(\|G_i \hat{u}\| = 0) > 0 \] and \( \Pr(0 < \|G_i \hat{u}\| < \theta_1) = 0 \)  
  Ch. 6, p. 54
Illustration

Original differences $U_i - U_{i+1}$ i.i.d. $\sim f(t) \propto e^{-\lambda \varphi(t)}$ on $[-\gamma, \gamma]$, $\varphi(t) = \frac{\alpha |t|}{1+\alpha |t|}$

Original $u_o$ (—) by $f$ for $\alpha = 10$, $\lambda = 1$, $\gamma = 4$

Data $v = u_o + n (\cdots)$, $N \sim N(0, \sigma^2 I)$, $\sigma = 5$.

The true MAP $\hat{u}$ (—), $\beta = 2\sigma^2 \lambda$ versus the original $u_o (\cdots)$.

Knowing the true distributions, with the true parameters, is not enough.

Combining models remains an open problem
Knowledge on the features of the minimizers enables new energies yielding appropriate solutions to be conceived.

“‘We’re in Act I of a digital revolution.’”

Jay Cassidy (film editor at Mathematical Technologies Inc.)

Thank you!

10 Some References
10 Some References


